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ON THE TELEGRAPH EQUATION WITH A SMALL PARAMETER

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Introduction

In this paper we will be concerned with the *telegraph equation*

$$a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} + cu = \frac{\partial^2 u}{\partial x^2}, \tag{T}$$

where $a > 0, b > 0$ and $c \geq 0$. We will be interested in studying the behaviour of solutions of this equation for $t \geq 0$ in the case where the coefficient a is small. In particular, our aim will be to estimate the difference between solutions of (T) and solutions of the *diffusion equation*

$$b\frac{\partial u}{\partial t} + cu = \frac{\partial^2 u}{\partial x^2}. \tag{D}$$

Such estimates have been studied by a number of authors, including M. Zlámal [11–13], J. Kisyński [2,3], J. Smoller [7], J. Nedoma [6], and M. Sova [9]. The paper by Zlámal [11] was inspired by a very specific technical problem. The question of estimating the difference between solutions of (T) and (D) has also been considered in the context of neutron transport, see A.M. Weinberg and E.P. Wigner [10, p. 235] and W. Baran [1].

It turned out that it is fruitful to treat (T) as a particular case of a certain type of differential equations in Banach spaces. For, such an approach allows uniform, systematic treatment of a number of boundary-value problems for (T) and obtaining in all these cases the same estimates in terms of the norms of underlying Banach spaces in which the problem is well-posed. We stress here that boundary-value problems that fit into our framework may include the solution u and its spatial derivative $\frac{\partial u}{\partial x}$, but not its time derivative $\frac{\partial u}{\partial t}$.

The difference between the results we present here and those obtained in [3] lies in the fact that now we are able to consider the case of non-zero c. This is important for applications e.g. in neuron transport theory. To treat $c > 0$, we introduce a new operator-valued function which we denote by $V(t, a, b, c, A)$. This function will critically intervene in what we will call the comparison theorem (see [3, p. 372] and Section 3 in the present paper).

1. Abstract Telegraph Equation

Let E_0 be a Banach space, let $\xi(t), -\infty < t < \infty$, be a strongly continuous cosine operator function with values in the space $\mathcal{L}(E_0, E_0)$, and let A be the infinitesimal generator of this cosine operator function, see M. Sova [8] or J. Kisyński [4]. Let

$$E_1 = \{u \in E_0 : \text{the function } (-\infty, \infty) \ni t \mapsto \xi(t)u \in E_0 \text{ is of class } C^1 \text{ in the norm of } E_0\}.$$

When equipped with the norm

$$\|u\|_{E_1} = \|u\|_{E_0} + \sup_{0\le 1\le t} \left\| \frac{d\xi(t)u}{dt} \right\|_{E_0},$$

E_1 is a Banach space; see [4, p. 98]. Let us consider the Cartesian product $E_1 \times E_0$ and let us agree to denote its elements as column-vectors $\binom{u}{v}, u \in E_1, v \in E_0$. Let k be a positive constant. Then the operators

$$G(t) = \begin{pmatrix} \xi(kt) & \int\limits_0^t \xi(k\tau)\,\mathrm{d}\tau \\ \frac{\mathrm{d}\xi(kt)}{\mathrm{d}t} & \xi(kt) \end{pmatrix}, \quad -\infty < t < \infty,$$

belong to $\mathcal{L}(E_1 \times E_0, E_1 \times E_0)$ and form a one-parameter strongly continuous group with generator

$$B = \begin{pmatrix} 0 & 1 \\ k^2 A & 0 \end{pmatrix}, \quad D(B) = D(A) \times E_1;$$

see [4, p. 98]. Now, let a, b and c be fixed scalars with $a > 0$. By the Dyson–Phillips bounded perturbation theorem, the operator

$$B_{a,b,c} = \begin{pmatrix} 0 & 1 \\ a^{-1}(A-c) & -a^{-1}b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a^{-1}A & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ a^{-1}c & a^{-1}b \end{pmatrix}$$

generates a strongly continuous one parameter group of operators in $E_1 \times E_0$. Let us express the operators in this group in the form of operator-valued matrices

$$\exp(tB_{a,b,c}) = \begin{pmatrix} S_{00}(t,a,b,c,A) & S_{01}(t,a,b,c,A) \\ S_{10}(t,a,b,c,A) & S_{11}(t,a,b,c,A) \end{pmatrix}.$$

Then the $S_{ij}(t,a,b,c,A)$ are strongly continuous functions of t with values in $\mathcal{L}(E_{1-j}, E_{1-i})$. It is easy to see that the Cauchy problem

$$\begin{cases} a\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\dfrac{\mathrm{d}u}{\mathrm{d}t} + cu = Au, \quad -\infty < t < \infty, \\ u(0) = u_0, \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(0) = u_1 \end{cases} \tag{T*}$$

has, for any pair of initial conditions $u_0 \in \mathcal{D}(A)$ and $u_1 \in E_1$, a $C^2(-\infty, \infty; E_0)$-class solution $u(t)$ with values in $\mathcal{D}(A)$, given by the formula

$$u(t) = S_{00}(t,a,b,c,A)u_0 + S_{01}(t,a,b,c,A)u_1.$$

This solution is unique in the class $C^2(0,T;E_0)$ on each interval $(0,T)$. To prove this, suppose that $v(t)$ is another solution of (T^*). Then, for any arbitrarily fixed $t \in (0,T)$,

$$v(t) - u(t) = \left[S_{00}(t-\tau)v(\tau) + S_{01}(t-\tau)\frac{\mathrm{d}v(\tau)}{\mathrm{d}\tau} \right]\Bigg|_{\tau=0}^{\tau=t}.$$

Using formulae (3) and (4), given in Section 3, for the derivatives of S_{00} and S_{11}, one may show that the derivative with respect to τ of the expression in the brackets is zero, and this implies that $v(t) = u(t)$. Another proof of uniqueness may be obtained by eliminating the first derivative in the differential equation from (T^*) with the substitution $u(t) = \mathrm{e}^{-\frac{b}{2a}t}v(t)$ and thereby reducing the problem to the case considered in [4, p. 96].

The operator-valued functions S_{ij} were build from the cosine function $\xi(t)$ via the group $G(t)$ by applying the Dyson–Phillips bounded perturbation theorem. One can show, and this will be done below, that if $b^2 > 4ac$ (a condition which henceforth will be tacitly assumed), then

$$S_{01}(t,a,b,c,A) = \mathrm{e}^{-\frac{b}{2a}t} \int_0^t J_0\left(i\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2} \right) \xi\left(\frac{\tau}{\sqrt{a}} \right) \mathrm{d}\tau, \tag{1}$$

where $\Delta = b^2 - 4ac$ and $J_0(ix)$ is the Bessel function of type zero with purely imaginary argument, given by the series

$$J_0(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2k}}{(k!)^2}.$$

An immediate consequence of equality (1) and formula (4) from Section 3 is

$$S_{00}(t,a,b,c,A) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{b}{a} \right) S_{01}(t,a,b,c,A) =$$
$$= \mathrm{e}^{-\frac{b}{2a}t}\xi\left(\frac{t}{2a}\right) + \mathrm{e}^{-\frac{b}{2a}t} \int_0^t \left(\frac{\partial}{\partial t} + \frac{b}{2a} \right) J_0\left(i\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2} \right) \xi\left(\frac{\tau}{\sqrt{a}} \right) \mathrm{d}\tau.$$

Hence, we see that the solution to (T^*) is given by the formula

$$u(t) = \mathrm{e}^{-\frac{b}{2a}t}\xi\left(\frac{t}{\sqrt{a}}\right)u_0 + \mathrm{e}^{-\frac{b}{2a}t} \int_0^t \left(\frac{\partial}{\partial t} + \frac{b}{2a} \right) J_0\left(i\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2} \right) \xi\left(\frac{\tau}{\sqrt{a}} \right) u_0\,\mathrm{d}\tau +$$
$$+ \mathrm{e}^{-\frac{b}{2a}t} \int_0^t J_0\left(i\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2} \right) \xi\left(\frac{\tau}{\sqrt{a}} \right) u_1\,\mathrm{d}\tau.$$

We now prove (1) by verifying that the right-hand side of the formula is a solution of an appropriate uniquely solvable Cauchy problem. For $u_1 \in E_1$, we have

$$\begin{cases} \left(a\frac{\mathrm{d}^2}{\mathrm{d}t^2} + b\frac{\mathrm{d}}{\mathrm{d}t} + c \right) S_{01}(t,a,b,c,A)u_1 = AS_{01}(t,a,b,c,A)u_1, \\ S_{01}(0,a,b,c,A)u_1 = 0, \\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S_{01}(t,a,b,c,A)u_1 = u_1, \end{cases} \tag{$*$}$$

and, as explained above, this system has a unique solution. Let

$$K(t,\tau) = \mathrm{e}^{-\frac{b}{2a}t} J_0\left(i\frac{\sqrt{\Delta}}{2a}\sqrt{\tau^2 - t^2}\right).$$

Now, all we need is to check that the integral

$$\tilde{S}_{01}(t)u_1 = \int_0^t K(t,\tau)\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau$$

satisfies $(*)$. The condition $\tilde{S}_{01}(0)u_1 = 0$ is obviously met. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{S}_{01}(t)u_1 = K(t,t)\xi\left(\frac{t}{\sqrt{a}}\right)u_1 + \int_0^t \frac{\partial K}{\partial t}(t,\tau)\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau =$$
$$= \mathrm{e}^{-\frac{b}{2a}t}\xi\left(\frac{t}{\sqrt{a}}\right)u_1 + \int_0^t \frac{\partial K}{\partial t}(t,\tau)\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau,$$

and so $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S_{01}(t)u_1 = u_1$. We are thus left with checking that the differential equation in system $(*)$ holds. To this end, we calculate

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{S}_{01}(t)u_1 = -\frac{b}{2a}\mathrm{e}^{-\frac{b}{2a}t}\xi\left(\frac{t}{\sqrt{a}}\right)u_1 + \mathrm{e}^{-\frac{b}{2a}t}\frac{\mathrm{d}}{\mathrm{d}t}\xi\left(\frac{t}{\sqrt{a}}\right)u_1 +$$
$$+ \frac{\partial K}{\partial t}(t,t)\xi\left(\frac{t}{\sqrt{a}}\right)u_1 + \int_0^t \frac{\partial^2 K}{\partial t^2}(t,t)\xi\left(\frac{t}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau$$

and find that

$$\left(a\frac{\mathrm{d}^2}{\mathrm{d}t^2}+b\frac{\mathrm{d}}{\mathrm{d}t}+c-A\right)\tilde{S}_{01}(t)u_1 = \left[\frac{b}{2}\mathrm{e}^{-\frac{b}{2a}t}+\frac{\partial K}{\partial t}(t,t)\right]\xi\left(\frac{t}{\sqrt{a}}\right)u_1+a\mathrm{e}^{-\frac{b}{2a}t}\frac{\mathrm{d}}{\mathrm{d}t}\xi\left(\frac{t}{\sqrt{a}}\right)u_1 +$$
$$+\int_0^t \left(a\frac{\partial^2}{\partial t^2}+b\frac{\partial}{\partial \tau}+c\right)K(t,\tau)\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}t - \int_0^t K(t,\tau)A\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau.$$

The last integral in this formula can be transformed by means of integration by parts as follows:

$$\int_0^t K(t,\tau)A\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau = a\int_0^t K(t,\tau)\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau =$$
$$= aK(t,\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\Big|_{\tau=0}^{\tau=t} - a\int_0^t \frac{\partial K}{\partial \tau}(t,\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau =$$
$$= aK(t,t)\frac{\mathrm{d}}{\mathrm{d}t}\xi\left(\frac{t}{\sqrt{a}}\right)u_1 - a\frac{\partial K}{\partial \tau}(t,\tau)\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\Big|_{\tau=0}^{\tau=t} + a\int_0^t \frac{\partial^2 K}{\partial \tau^2}(t,\tau)\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau.$$

Since $\frac{\partial K}{\partial \tau}(t,0)=0$, we thus obtain

$$\left(a\frac{\mathrm{d}^2}{\mathrm{d}t^2}+b\frac{\mathrm{d}}{\mathrm{d}t}+c-A\right)\tilde{S}_{01}(t)u_1 = a\left[\frac{b}{2a}\mathrm{e}^{-\frac{b}{2a}t}+\frac{\partial K}{\partial t}(t,t)+\frac{\partial K}{\partial \tau}(t,t)\right]\xi\left(\frac{t}{\sqrt{a}}\right)u_1 +$$
$$+\int\limits_0^t\left[a\frac{\partial^2 K}{\partial t^2}(t,\tau)-a\frac{\partial^2 K}{\partial \tau^2}(t,\tau)+b\frac{\partial K}{\partial t}(t,\tau)+cK(t,\tau)\right]\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau =$$
$$=\int\limits_0^t\left[a\frac{\partial^2 K}{\partial t^2}(t,\tau)-a\frac{\partial^2 K}{\partial \tau^2}(t,\tau)+b\frac{\partial K}{\partial t}(t,\tau)+cK(t,\tau)\right]\xi\left(\frac{\tau}{\sqrt{a}}\right)u_1\,\mathrm{d}\tau.$$

It is now clear that the proof will be complete once we show that

$$a\frac{\partial^2 K}{\partial t^2}(t,\tau)-a\frac{\partial^2 K}{\partial t^2}(t,\tau)+b\frac{\partial K}{\partial t}(t,\tau)+cK(t,\tau)=0 \tag{$\substack{*\\*}$}$$

for $0\le\tau\le t$. Let $K_0(t,\tau)=J_0\left(i\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2}\right)$. Then we have $K(t,\tau)=\mathrm{e}^{-\frac{b}{2a}t}K_0(t,\tau)$, and equation $(\substack{*\\*})$ takes the equivalent form

$$\left[a\left(\frac{\partial}{\partial t}-\frac{b}{2a}\right)^2-a\frac{\partial^2}{\partial\tau^2}+b\left(\frac{\partial}{\partial t}-\frac{b}{2a}\right)+c\right]K_0(t,\tau)=0,$$

which is the same as

$$a\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial \tau^2}-\frac{b^2-4ac}{4a^2}\right]K_0(t,\tau)=0,$$

or

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial \tau^2}-\frac{\Delta}{4a^2}\right)K_0(t,\tau)=0. \tag{$\substack{*\\**}$}$$

Given that

$$\frac{\partial^2 K_0}{\partial t^2}(t,\tau)=\left(\frac{\sqrt{\Delta}}{2a}\frac{t}{\sqrt{t^2-\tau^2}}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\bigg|_{x=\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2}}J_0(ix)+$$
$$+\frac{\sqrt{\Delta}}{2a}\left(\frac{1}{\sqrt{t^2-\tau^2}}-\frac{t^2}{\left(\sqrt{t^2-\tau^2}\right)^3}\right)\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2}}J_0(ix)$$

and

$$\frac{\partial^2 K_0}{\partial \tau^2}(t,\tau)=\left(\frac{\sqrt{\Delta}}{2a}\frac{\tau}{\sqrt{t^2-\tau^2}}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\bigg|_{x=\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2}}J_0(ix)+$$
$$+\frac{\sqrt{\Delta}}{2a}\left(\frac{-1}{\sqrt{t^2-\tau^2}}-\frac{\tau^2}{\left(\sqrt{t^2-\tau^2}\right)^3}\right)\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=\frac{\sqrt{\Delta}}{2a}\sqrt{t^2-\tau^2}}J_0(ix),$$

we see that equation $(\substack{*\\**})$ is equivalent to

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}-1\right)J_0(ix)=0,$$

and that the latter equation holds can be checked immediately by differentiating the series

$$J_0(ix) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2k}}{(k!)^2}$$

term by term.

2. Abstract Diffusion Equation

Along with the second order problem (T*) we consider the initial value problem of order one

$$\begin{cases} b\dfrac{\mathrm{d}u}{\mathrm{d}t} + cu = Au, \quad t \geq 0, \\ \qquad u(0) = u_0. \end{cases} \tag{D*}$$

The solution of this problem is

$$u(t) = \mathrm{e}^{-\frac{c}{b}t} \exp\left(\frac{t}{b}A\right),$$

where $\exp(tA)$, $t \geq 0$, is the one-parameter semigroup of operators in the space E_0 generated by A. This semigroup may be simply expressed in terms of the cosine family from the previous section, namely,

$$\exp(tA) = \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\frac{\tau^2}{4t}} \xi(\tau)\,\mathrm{d}\tau, \quad t > 0; \tag{2}$$

see J. Kisyński [5, p. 9].

3. Reducing the Problem of Estimating the Difference Between Solutions of Problems (T*) and (D*) to a Comparison Theorem

From $\exp(tB_{a,b,c})D(B_{a,b,c}) = D(B_{a,b,c})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tB_{a,b,c})\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = B_{a,b,c}\exp(tB_{a,b,c})\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \exp(tB_{a,b,c})B_{a,b,c}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

with the latter equality holding for $u_0 \in \mathcal{D}(A)$ and $u_1 \in E_1$, and taking the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} S_{00}(t) & S_{01}(t) \\ S_{10}(t) & S_{11}(t) \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a^{-1}(A-c) & -a^{-1}b \end{pmatrix}\begin{pmatrix} S_{00}(t) & S_{01}(t) \\ S_{10}(t) & S_{11}(t) \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} =$$
$$= \begin{pmatrix} S_{00}(t) & S_{01}(t) \\ S_{10}(t) & S_{11}(t) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ a^{-1}(A-c) & -a^{-1}b \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

we deduce the inclusions $S_{00}D(A) \subset D(A)$, $S_{01}E_1 \subset D(A)$, $S_{10}D(A) \subset E_1$ and $S_{11}E_1 \subset E_1$, and equalities

$$\frac{\mathrm{d}S_{00}u_0}{\mathrm{d}t} = S_{10}u_0 = S_{01}a^{-1}(A-c)u_0, \tag{3}$$
$$\frac{\mathrm{d}S_{01}u_1}{\mathrm{d}t} = S_{11}u_1 = S_{00}u_1 - \frac{b}{a}S_{01}u_1, \tag{4}$$
$$\frac{\mathrm{d}S_{10}u_0}{\mathrm{d}t} = a^{-1}(A-c)S_{00}u_0 - \frac{b}{a}S_{10}u_0 = S_{11}a^{-1}(A-c)u_0, \tag{5}$$
$$\frac{\mathrm{d}S_{11}u_1}{\mathrm{d}t} = a^{-1}(A-c)S_{01}u_1 - \frac{b}{a}S_{11}u_1 = S_{10}u_1 - \frac{b}{a}S_{11}u_1. \tag{6}$$

If $u_0 \in \mathcal{D}(A)$, then it follows from (3) and (6) applied with $u_1 = u_0$ that

$$S_{01}a^{-1}(A-c)u_0 = a^{-1}(A-c)S_{01}u_0 = S_{10}u_0. \tag{7}$$

On the other hand, by (4), (5) and (7) it is clear that if S_{00} is extended by continuity to an operator in $\mathcal{L}(E_0, E_0)$, which is possible by (4), then, for $u_0 \in D(A)$, we have

$$S_{00}Au_0 = AS_{00}u_0. \tag{8}$$

By (3) and (5), we see that for $u_0 \in D(A)$ the function $u(t) = S_{00}(t)u_0$ satisfies the equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = b^{-1}(A-c)u(t) - S_{11}(t)b^{-1}(A-c)u_0.$$

Since $u(0) = u_0$, it follows that

$$S_{00}(t)u_0 = \exp\left(\frac{t}{b}(A-c)\right)u_0 - \int\limits_0^t \exp\left(\frac{t-\tau}{b}(A-c)\right)S_{11}(\tau)b^{-1}(A-c)u_0\,\mathrm{d}\tau. \tag{9}$$

To simplify notation, let

$$U(t,a,b,c,A) = \int\limits_0^t \exp\left(\frac{t-\tau}{b}(A-c)\right)S_{11}(\tau,a,b,c,A)\,\mathrm{d}\tau.$$

In view of (9), the difference between the solution

$$u_a(t) = S_{00}(t,a,b,c,A)u_0 + S_{01}(t,a,b,c,A)u_1$$

to problem (T*) and the solution

$$u(t) = \exp\left(\frac{t}{b}(A-c)\right)u_0$$

to problem (D*) satisfies the estimate

$$\|u_a(t) - u(t)\|_{E_0} \le \left\|U(t,a,b,c,A)b^{-1}(A-c)u_0\right\|_{E_0} + \|S_{01}(t,a,b,c,A)u_1\|_{E_0} \tag{10}$$

for $u_0 \in D(A)$, $u_1 \in E_1$ and $t \geq 0$. We will also obtain a similar estimate for the difference $\frac{\mathrm{d}u_0(t)}{\mathrm{d}t} - \frac{\mathrm{d}u(t)}{\mathrm{d}t}$. Since $S_{11}(0)u_1 = u_1$, (6) implies that

$$S_{11}(t)u_1 = \mathrm{e}^{-\frac{b}{a}t}u_1 + \int\limits_0^t \mathrm{e}^{-\frac{b}{a}(t-\tau)} S_{10}(\tau)u_1 \,\mathrm{d}\tau.$$

If $u_1 \in D(A)$, then by (3) we may write

$$S_{11}(t)u_1 = \mathrm{e}^{-\frac{b}{a}t}u_1 + \frac{b}{a}\int\limits_0^t \mathrm{e}^{-\frac{b}{a}(t-\tau)} S_{01}(\tau)b^{-1}(A-c)u_1 \,\mathrm{d}\tau. \tag{11}$$

If $u_0 \in D(A^2)$ and $u_1 \in D(A)$, then, by (3) and (4),

$$\frac{\mathrm{d}u_a(t)}{\mathrm{d}t} - \frac{\mathrm{d}u(t)}{\mathrm{d}t} - \mathrm{e}^{-\frac{b}{a}t}(u_1 - b^{-1}(A-c)u_0)$$

equals

$$S_{10}(t)u_0 + S_{11}(t)b^{-1}(A-c)u_0 - \exp\left(\frac{t}{b}(A-c)\right) b^{-1}(A-c)u_0 + \\ + \left(S_{11}(t) - \mathrm{e}^{-\frac{b}{a}t}\right)\left(u_1 - b^{-1}(A-c)u_0\right)$$

which in turn, given that, by (5) and (8),

$$S_{10}(t)u_0 + S_{11}(t)b^{-1}(A-c)u_0 = b^{-1}(A-c)S_{00}(t)u_0 = S_{00}(t)b^{-1}(A-c)Bu_0,$$

is equal to

$$S_{00}(t)b^{-1}(A-c)u_0 - \exp\left(\frac{t}{b}(A-c)\right) b^{-1}(A-c)u_0 + \left(S_{11}(t) - \mathrm{e}^{-\frac{b}{a}t}\right)\left(u_1 - b^{-1}(A-c)u_0\right),$$

and this, in view of (9) and (11), finally equals

$$-U(t,a,b,c,A)\left[b^{-1}(A-c)\right]^2 u_0 + \\ + \frac{b}{a}\int\limits_0^t \mathrm{e}^{-\frac{b}{a}(t-\tau)} S_{01}(\tau,a,b,c,A)b^{-1}(A-c)\left[u_1 - b^{-1}(A-c)u_0\right]\,\mathrm{d}\tau.$$

Hence, under the assumption that $u_0 \in D(A^2)$ and $u_1 \in D(A)$, we obtain the following estimate:

$$\begin{gathered}\left\|\tfrac{\mathrm{d}u_a(t)}{\mathrm{d}t} - \tfrac{\mathrm{d}u(t)}{\mathrm{d}t} - \mathrm{e}^{-\frac{b}{a}t}(u_1 - b^{-1}(A-c)u_0)\right\|_{E_0} \leq \\ \leq \left\|U(t,a,b,c,A)\left[b^{-1}(A-c)\right]^2 u_0\right\|_{E_0} + \\ + \sup_{0\leq\tau\leq t}\left\|S_{01}(\tau,a,b,c,A)b^{-1}(A-c)\left[u_1 - b^{-1}(A-c)u_0\right]\right\|_{E_0}.\end{gathered} \tag{12}$$

In view of inequalities (10) and (12), the problem of estimating the difference $u_a(t) - u(t)$ reduces to estimating the norms of operator-valued functions $U(t,a,b,c,A)$ and

$S_{11}(t,a,b,c,A)$. The case of U will not be treated directly, but will be tackled by means of the auxiliary function V defined by

$$V(t,a,b,c,A) = U(t,a,b,c,A) + \frac{b-\sqrt{\Delta}}{2a}\int\limits_0^t U(\tau,a,b,c,A)\,\mathrm{d}\tau. \tag{13}$$

Viewing (13) as a Volterra-type equation with U as an unknown function, we easily deduce that

$$U(t,a,b,c,A) = V(t,a,b,c,A) + \frac{\sqrt{\Delta}-b}{2a}\int\limits_0^t \mathrm{e}^{\frac{\sqrt{\Delta}-b}{2a}(t-\tau)} V(\tau,a,b,c,A)\,\mathrm{d}\tau. \tag{14}$$

Since $\frac{\sqrt{\Delta}-b}{2a} < 0$, (14) implies immediately that

$$\|U(t,a,b,c,A)\| \le 2 \sup_{0\le\tau\le t} \|V(t,a,b,c,A)\|.$$

However, this inequality will not be used in the context of the telegraph equation $(*)$ which is of main interest to us, as in this case other, more subtle estimates are available and will be exploited that result from special properties of V.

The following theorem is the key to estimating the norms of operator-valued functions $V(t,a,b,c,A)$ and $S_{01}(t,a,b,c,A)$.

Comparison Theorem. *Let $a>0$, $b>0$ and $c\ge 0$ be such that $\Delta = b^2-4ac>0$, and let $k\ge 0$. Let $S_{ij}(t,a,b,c,k^2)$ be the real functions defined via the equality*

$$\begin{pmatrix} S_{00}(t,a,b,c,k^2) & S_{01}(t,a,b,c,k^2) \\ S_{10}(t,a,b,c,k^2) & S_{11}(t,a,b,c,k^2)\end{pmatrix} = \exp\left(t\begin{pmatrix} 0 & 1 \\ \dfrac{k^2-c}{a} & -\frac{b}{a}\end{pmatrix}\right)$$

and let (compare (13))

$$V(t,a,b,c,k^2) := \mathrm{e}^{\frac{k^2-c}{b}t} * S_{11}(t,a,b,c,k^2) + \frac{b-\sqrt{\Delta}}{2a} * \mathrm{e}^{\frac{k^2-c}{b}t} * S_{11}(t,a,b,c,k^2),$$

where $*$ denotes convolution on the half-line $t\ge 0$. If

$$\|\xi(t)\|_{L(E_0,E_0)} \le M\cosh(kt), \qquad -\infty<t<\infty, \tag{15}$$

with $M = const \ge 1$, then

$$\left\|\exp\left(\frac{t}{b}(A-c)\right)\right\|_{L(E_0,E_0)} \le M\exp\left(\frac{t}{b}(k^2-c)\right), \qquad t\ge 0, \tag{16}$$

$$\|S_{00}(t,a,b,c,A)\|_{L(E_0,E_0)} \le M\,S_{00}(t,a,b,c,k^2), \qquad t\ge 0, \tag{17}$$

$$\|S_{01}(t,a,b,c,A)\|_{L(E_0,E_0)} \le M\,S_{01}(t,a,b,c,k^2), \qquad t\ge 0, \tag{18}$$

and

$$\|V(t,a,b,c,A)\|_{L(E_0,E_0)} \le M\,V(t,a,b,c,k^2), \qquad t\ge 0. \tag{19}$$

4. Proof of the Comparison Theorem

By (2), we have

$$\|\exp(tA)\| \le \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\frac{\tau^2}{4t}} \|\xi(\tau)\| \, \mathrm{d}\tau \le M \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\frac{\tau^2}{4t}} \cosh(kt) \, \mathrm{d}\tau = M \exp(tk^2),$$

and so

$$\left\| \exp\left(\frac{t}{b}(A-c)\right) \right\| \le \mathrm{e}^{-\frac{c}{b}t} \|\exp(tA)\| \le M \mathrm{e}^{-\frac{c}{b}t} \mathrm{e}^{\frac{t}{b}k^2} = M \mathrm{e}^{\frac{k^2-c}{b}t}.$$

Similarly, by (1),

$$\|S_{01}(t,a,b,c,A)\| \le \int\limits_0^t K(t,\tau) \left\| \xi\left(\frac{\tau}{\sqrt{a}}\right) \right\| \mathrm{d}\tau \le$$

$$\le M \int\limits_0^t K(t,\tau) \cosh\left(\frac{k\tau}{\sqrt{a}}\right) \mathrm{d}\tau = M S_{01}(t,a,b,c,k^2),$$

because the kernel $K(t,\tau) = \mathrm{e}^{-\frac{b}{2a}t} J_0\left(i\frac{\sqrt{\Delta}}{2a}\sqrt{\tau^2 - t^2}\right)$ is non-negative for $0 \le \tau \le t$. Inequalities (16) and (18) are thus proved. Inequality (17) is established in a similar manner. The proof of (19) is more complicated. According to a well-known result from the theory of semigroups of operators, there is a constant λ_0 such that for $\lambda > \lambda_0$ we have

$$b\left(b\lambda + c - A\right)^{-1} = \left(\lambda - b^{-1}(A-c)\right)^{-1} = \int\limits_0^{\infty} \mathrm{e}^{-\lambda t} \exp\left(\frac{t}{b}(A-c)\right) \mathrm{d}t \tag{20}$$

and

$$\left(\lambda - B_{a,b,c}\right)^{-1} = \int\limits_0^{\infty} \mathrm{e}^{-\lambda t} \exp\left(tB_{a,b,c}\right) \mathrm{d}t. \tag{21}$$

Since (15) implies that

$$\lambda(\lambda^2 - A)^{-1} = \int\limits_0^{\infty} \mathrm{e}^{-\lambda t} \xi(t) \, \mathrm{d}t \quad \text{for } \lambda > k, \tag{22}$$

it follows that for λ so large that $a\lambda^2 + b\lambda + c > k^2$ we have

$$(\lambda - B_{a,b,c})^{-1} = \left(\lambda - \begin{pmatrix} 0 & 1 \\ a^{-1}(A-c) & -a^{-1}b \end{pmatrix}\right)^{-1} =$$

$$= \begin{pmatrix} (a\lambda + b)(a\lambda^2 + b\lambda + c - A)^{-1} & a(a\lambda^2 + b\lambda + c - A)^{-1} \\ (A-c)(a\lambda^2 + b\lambda + c - A)^{-1} & a\lambda(a\lambda^2 + b\lambda + c - A)^{-1} \end{pmatrix}.$$

This together with (21) implies the existence of a constant λ_1 such that

$$\int\limits_0^{\infty} \mathrm{e}^{-\lambda t} S_{11}(t,a,b,c,A) = a\lambda(a\lambda^2 + b\lambda + c - A)^{-1} \quad \text{for } \lambda > \lambda_1. \tag{23}$$

Since the Laplace transform of the convolution of two functions (on the half-line $0 \le t < \infty$) is the product of the transforms of these functions, equality (13) written as

$$V(t,a,b,c,A) = \exp\left(\frac{t}{b}(A-c)\right) * S_{11}(t,a,b,c,A) + \\ + \frac{b-\sqrt{\Delta}}{2a} * \exp\left(\frac{t}{b}(A-c)\right) * S_{11}(t,a,b,c,A)$$

and combined with (20) and (23) implies that

$$\int_0^\infty \mathrm{e}^{-\lambda t} V(t,a,b,c,A)\,\mathrm{d}t = a\left(\lambda + \frac{b-\sqrt{\Delta}}{2a}\right)\left(\lambda - b^{-1}(A-c)\right)^{-1}(a\lambda^2 + b\lambda + c - A)^{-1}$$

for $\lambda > \lambda_2 = \max(\lambda_0, \lambda_1)$. Introducing the functions

$$g_1(\lambda) = \left(\lambda + \frac{c}{b}\right)^{\frac{1}{2}}, \\ g_2(\lambda) = (a\lambda^2 + b\lambda + c)^{\frac{1}{2}}, \\ h(\lambda) = a\frac{\lambda + \frac{b-\sqrt{\Delta}}{2a}}{\left(\lambda + \frac{c}{b}\right)^{\frac{1}{2}}\left(a\lambda^2 + b\lambda + c\right)^{\frac{1}{2}}},$$

we may rewrite the last identity as

$$\int_0^\infty \mathrm{e}^{-\lambda t} V(t,a,b,c,A)\,\mathrm{d}t = h(\lambda) g_1(\lambda)\left([g_1(\lambda)]^2 - A\right)^{-1} g_2(\lambda)\left([g_2(\lambda)]^2 - A\right)^{-1}$$

for $\lambda > \lambda_2$. But now, by (22), we see that

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t} V(t,a,b,c,A)\,\mathrm{d}t &= h(\lambda)\int_0^\infty \mathrm{e}^{-g_1(\lambda)t}\xi(t)\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-g_2(\lambda)s}\xi(s)\,\mathrm{d}s = \\ &= h(\lambda)\int_0^\infty\int_0^\infty \mathrm{e}^{-g_1(\lambda)t - g_2(\lambda)s}\frac{1}{2}[\xi(t+s) + \xi(t-s)]\,\mathrm{d}t\,\mathrm{d}s\end{aligned} \tag{24}$$

for $\lambda > \lambda_2$. Analogously,

$$\int_0^\infty \mathrm{e}^{-\lambda t} V(t,a,b,c,k^2)\,\mathrm{d}t = h(\lambda)\int_0^\infty\int_0^\infty \mathrm{e}^{-g_1(\lambda)t - g_2(\lambda)s}\frac{1}{2}[\cosh(t+s) + \cosh(t-s)]\,\mathrm{d}t\,\mathrm{d}s. \tag{25}$$

Thus, for each $u \in E_0$ such that $\|u\| = 1$ and each bounded linear functional f on E_0 such that $\|f\| = 1$, we have

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t}\left[M\,V(t,a,b,c,k^2) - \langle f, V(t,a,b,c,A)u\rangle\right]\,\mathrm{d}t = \\ = h(\lambda)\int_0^\infty\int_0^\infty \mathrm{e}^{-g_1(\lambda)t - g_2(\lambda)s}\Psi(t,s)\,\mathrm{d}t\,\mathrm{d}s\end{aligned} \tag{26}$$

for $\lambda > \lambda_2$, while, by (15), the function

$$\Psi(t,s) = \frac{1}{2}M[\cosh(t+s) + \cosh(t-s)] - \frac{1}{2}\langle f, [\xi(t+s) + \xi(t-s)]u\rangle$$

is non-negative. Let

$$F(\lambda) = h(\lambda)\int\limits_0^\infty\int\limits_0^\infty \mathrm{e}^{-g_1(\lambda)t - g_2(\lambda)s}\Psi(t,s)\,\mathrm{d}t\,\mathrm{d}s$$

and let us assume for now that

the function $F(-\lambda)$ is completely monotone in the interval $-\infty < \lambda < -\lambda_2$, i.e., that $\frac{\mathrm{d}^n F(-\lambda)}{\mathrm{d}\lambda^n} > 0$ for all $n = 0, 1, \ldots$ and $\lambda \in (-\infty, -\lambda_2)$. (27)

It follows (26) and (27), by virtue of the Post–Widder formula for inversion of the Laplace transform, that

$$M\,V(t,a,b,c,k^2) - \langle f, V(t,a,b,c,A)u\rangle = \lim_{n\to\infty}\frac{(-1)^n}{(n-1)!}\left(\frac{n}{t}\right)^n \left.\frac{\mathrm{d}^{n-1}F(\lambda)}{\mathrm{d}\lambda^{n-1}}\right|_{\lambda=\frac{n}{t}} \geq 0$$

for each $t > 0$. Since the only restriction imposed on u and f was $\|u\| = \|f\| = 1$, this results in inequality (19).

The idea of exploiting the Post–Widder formula to obtain estimates as above is taken from the paper by M. Sova [9], who was the first to study the asymptotic behaviour, as $a \to 0$, of solutions of initial-value problem of type (T*) in a non-Hilbert space. The idea to use completely monotone functions is due to the author of the present paper; this approach simplifies the analysis and sharpens the estimates.

We now present the proof of (27). It suffices to show that

$$\frac{\mathrm{d}^n[-g_1(-\lambda)]}{\mathrm{d}\lambda^n} > 0 \qquad \text{for } n = 0, 1, \ldots \text{ and } \lambda \in (-\infty, -\lambda_2), \tag{a}$$

$$\frac{\mathrm{d}^n[-g_2(-\lambda)]}{\mathrm{d}\lambda^n} > 0 \qquad \text{for } n = 0, 1, \ldots \text{ and } \lambda \in (-\infty, -\lambda_2), \tag{b}$$

and

$$\frac{\mathrm{d}^n h(-\lambda)}{\mathrm{d}\lambda^n} > 0 \qquad \text{for } n = 0, 1, \ldots \text{ and } \lambda \in (-\infty, -\lambda_2). \tag{c}$$

That (a) is true follows immediately from the formula

$$\frac{\mathrm{d}^n[-g_1(-\lambda)]}{\mathrm{d}\lambda^n} = -\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\frac{c}{b} - \lambda\right)^{\frac{1}{2}} = \frac{1}{2}\frac{1}{2}\frac{3}{2}\cdots\frac{2n-3}{2}\left(\frac{c}{b} - \lambda\right)^{-\frac{2n-1}{2}}.$$

For the proof of (b) we calculate, for $\lambda < 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[-g_2(-\lambda)] = -\frac{\mathrm{d}}{\mathrm{d}\lambda}(a\lambda^2 - b\lambda + c)^{\frac{1}{2}} = \left(\frac{b}{2} - a\lambda\right)\left(a\lambda^2 - b\lambda + c\right)^{-\frac{1}{2}} > 0$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}[-g_2(-\lambda)] = -a(a\lambda^2 - b\lambda + c)^{-\frac{1}{2}} + \left(\frac{b}{2} - a\lambda\right)^2 \left(a\lambda^2 - b\lambda + c\right)^{-\frac{3}{2}} =$$
$$= \frac{\Delta}{4}\left(a\lambda^2 - b\lambda + c\right)^{-\frac{3}{2}} = \frac{\Delta}{4}a^{-\frac{3}{2}}\left(\mu_1 - \lambda\right)^{-\frac{3}{2}}\left(\mu_2 - \lambda\right)^{-\frac{3}{2}} > 0,$$

where $\mu_1 = \frac{b-\sqrt{\Delta}}{2a} > 0$ and $\mu_2 = \frac{b+\sqrt{\Delta}}{2a} > 0$. Since

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(\mu_j - \lambda)^{-\frac{3}{2}} = \frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2k+1}{2}(\mu_j - \lambda)^{-\frac{2k+3}{2}}$$

for $k = 0, 1, \ldots, j = 1, 2$ and $\lambda < 0$, Leibniz's formula implies that

$$\frac{\mathrm{d}^{2+n}[-g_2(-\lambda)]}{\mathrm{d}\lambda^{2+n}} = \frac{\Delta}{4}a^{-\frac{3}{2}}\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k(\mu_1 - \lambda)^{-\frac{3}{2}}}{\mathrm{d}\lambda^k}\frac{\mathrm{d}^{n-k}(\mu_2 - \lambda)^{-\frac{3}{2}}}{\mathrm{d}\lambda^{n-k}} > 0$$

for $\lambda < 0$ and $n = 1, 2, \ldots$.

We are left with showing (c). To this end, we observe that

$$h(-\lambda) = a\frac{\mu_1 - \lambda}{\left(\frac{c}{a} - \lambda\right)^{\frac{1}{2}} a^{\frac{1}{2}}\left(\mu_1 - \lambda\right)^{\frac{1}{2}}\left(\mu_2 - \lambda\right)^{\frac{1}{2}}} = a^{\frac{1}{2}}\left(\frac{\mu_1 - \lambda}{\frac{c}{b} - \lambda}\right)^{\frac{1}{2}}\left(\mu_2 - \lambda\right)^{-\frac{1}{2}}.$$

Since

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(\mu_2 - \lambda)^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-1}{2}(\mu_2 - \lambda)^{-\frac{2k+1}{2}}$$

for $\lambda < 0$ and $k = 1, 2, \ldots$, Leibniz's formula shows that our task reduces to proving that

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\left(\frac{\mu_1 - \lambda}{\frac{c}{b} - \lambda}\right)^{\frac{1}{2}} > 0 \quad \text{for } \lambda < 0 \text{ and } k = 0, 1, \ldots . \tag{c*}$$

Inequality (c*) results from

$$D := \lambda_1 - \frac{c}{b} = \frac{b - \sqrt{\Delta}}{2a} - \frac{c}{b} \geq 0 \tag{28}$$

since $b^2 - 2ac \geq b\sqrt{\Delta}$; the latter condition may easily be checked by taking squares of both sides. By (28), we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mu_1 - \lambda}{\frac{c}{b} - \lambda}\right)^{\frac{1}{2}} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\frac{c}{b} + D - \lambda}{\frac{c}{b} - \lambda}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{\frac{c}{b} - \lambda}{\frac{c}{b} + D - \lambda}\right)^{\frac{1}{2}}\frac{D}{\left(\frac{c}{b} - \lambda\right)^2} =$$
$$= \frac{D}{2}\left(\frac{c}{b} + D - \lambda\right)^{-\frac{1}{2}}\left(\frac{c}{b} - \lambda\right)^{-\frac{3}{2}},$$

and thus we see that inequality (c*) is true for $k = 0, 1$. To prove it for $k = 2, 3, \ldots$, one needs to apply Leibniz's formula in a way similar to that used in our proof of (b). Hence, inequality (c) is seen to be true. The proof of the comparison theorem is thereby completed.

5. Consequences of the Comparison Theorem in the Case $k = 0$

We have

$$V(t,a,b,c,0) = \mathrm{e}^{-\frac{c}{b}t} * \left(S_{11}(t,a,b,c,0) + \frac{b-\sqrt{\Delta}}{2a} \int\limits_0^t S_{11}(\tau,a,b,c,0)\,\mathrm{d}\tau \right) =$$
$$= \mathrm{e}^{-\frac{c}{b}t} * \left(S_{11}(t,a,b,c,0) + \frac{b-\sqrt{\Delta}}{2a} S_{01}(t,a,b,c,0) \right) = \mathrm{e}^{-\frac{c}{b}t} * u(t),$$

where

$$u(t) = S_{11}(t,a,b,c,0) + \frac{b-\sqrt{\Delta}}{2a} S_{01}(t,a,b,c,0)$$

is the solution of the equation

$$au'' + bu' + cu = 0$$

with initial conditions

$$u(0) = 1,$$
$$u'(0) = S'_{11}(0) + \frac{b-\sqrt{\Delta}}{2a} = -\frac{b}{a} + \frac{b-\sqrt{\Delta}}{2a} = -\frac{b+\sqrt{\Delta}}{2a}.$$

Therefore we have

$$u(t) = \mathrm{e}^{-\frac{b+\sqrt{\Delta}}{2a}t}$$

and so

$$V(t,a,b,c,0) = \mathrm{e}^{-\frac{c}{b}t} \int\limits_0^t \mathrm{e}^{\left(\frac{c}{b} - \frac{b+\sqrt{\Delta}}{2a}\right)\tau}\,\mathrm{d}\tau = \mathrm{e}^{-\frac{c}{b}t} \int\limits_0^t \mathrm{e}^{-\left(D + \frac{\sqrt{\Delta}}{a}\right)\tau}\,\mathrm{d}\tau \le$$
$$\le \mathrm{e}^{-\frac{c}{b}t} \frac{1}{D + \frac{\sqrt{\Delta}}{a}} \le \frac{a}{\sqrt{\Delta}} \mathrm{e}^{-\frac{c}{b}t}.$$

Also

$$\int\limits_0^t \mathrm{e}^{\frac{\sqrt{\Delta}-b}{2a}(t-\tau)} V(\tau,a,b,c,0)\,\mathrm{d}\tau = \int\limits_0^t \mathrm{e}^{\frac{\sqrt{\Delta}-b}{2a}\tau} V(t-\tau,a,b,c,0)\,\mathrm{d}\tau \le$$
$$\le \frac{a}{\sqrt{\Delta}} \int\limits_0^t \mathrm{e}^{\frac{\sqrt{\Delta}-b}{2a}\tau} \mathrm{e}^{-\frac{c}{b}(t-\tau)}\,\mathrm{d}\tau = \frac{a}{\sqrt{\Delta}} \mathrm{e}^{-\frac{c}{b}t} \int\limits_0^t \mathrm{e}^{-D\tau}\,\mathrm{d}\tau.$$

Assume that inequality (15) holds with $k = 0$. Then, in view of the above, inequality (19) and equality (14), we obtain the estimate

$$\|U(t,a,b,c,A)\|_{L(E_0,E_0)} \le \mathrm{e}^{-\frac{c}{b}t} M \frac{a}{\sqrt{\Delta}} \left[1 + \frac{b-\sqrt{\Delta}}{2a} \int\limits_0^t \mathrm{e}^{-D\tau}\,\mathrm{d}\tau \right] \le$$
$$\le \mathrm{e}^{-\frac{c}{b}t} M \frac{a}{\sqrt{\Delta}} \left(1 + \frac{c}{\sqrt{\Delta}} t \right), \tag{29}$$

for we have $\frac{b-\sqrt{\Delta}}{2a} \le \frac{c}{\sqrt{\Delta}}$. Since $u(t) = S_{01}(t,a,b,c,0)$ is the solution of the equation $au'' + bu' + cu = 0$ satisfying the conditions $u(0) = 0$ and $u'(0) = 1$, we have

$S_{01}(t,a,b,c,0) = (\nu_1 - \nu_2)^{-1}(\mathrm{e}^{\nu_1 t} - \mathrm{e}^{\nu_2 t})$, where $\nu_1 = \frac{-b+\sqrt{\Delta}}{2a}$ and $\nu_2 = \frac{-b-\sqrt{\Delta}}{2a}$, which implies that $S_{01}(t,a,b,c,0) \le \frac{a}{\sqrt{\Delta}}$ for $t \ge 0$. In view of inequality (18), we thus have

$$\|S_{01}(t,a,b,c,A)\|_{L(E_0,E_0)} \le M\frac{a}{\sqrt{\Delta}} \quad \text{for } t \ge 0. \tag{30}$$

Combining together (10), (12), (29) and (30), we finally obtain the following estimates of the difference between the solution $u_a(t)$ of problem (T*) and the solution $u(t)$ of problem (D*):

(i) if $u_0 \in D(A)$ and $u_1 \in E_1$, then

$$\|u_a(t) - u(t)\|_{E_0} \le \frac{a}{\sqrt{\Delta}}M\left\{\mathrm{e}^{-\frac{c}{b}t}\left(1 + \frac{c}{\sqrt{\Delta}}t\right)\|b^{-1}(A-c)u_0\|_{E_0} + \|u_1\|_{E_0}\right\}$$

for $t \ge 0$;

(ii) and if, furthermore, $u_0 \in D(A^2)$ and $u_1 \in D(A)$, then also

$$\left\|\frac{\mathrm{d}u_a(t)}{\mathrm{d}t} - \frac{\mathrm{d}u(t)}{\mathrm{d}t} - \mathrm{e}^{-\frac{b}{a}t}(u_1 - b^{-1}(A-c)u_0)\right\|_{E_0} \le$$

$$\le \frac{a}{\sqrt{\Delta}}M\left\{\mathrm{e}^{-\frac{c}{b}t}\left(1 + \frac{c}{\sqrt{\Delta}}t\right)\|[b^{-1}(A-c)]^2u_0\|_{E_0} + \|b^{-1}(A-c)[u_1 - b^{-1}(A-c)u_0]\|_{E_0}\right\}$$

for $t \ge 0$.

For $c = 0$ these inequalities are identical to those obtained in the paper [3].

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