**MSC 47A68** **DOI: 10.14529/mmp180408**

# ALGORITHM OF POLYNOMIAL FACTORIZATION AND ITS IMPLEMENTATION IN MAPLE

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In the work we propose an algorithm for a Wiener–Hopf factorization of scalar polynomials. The algorithm based on notions of indices and essential polynomials allows to find the factorization factors of the polynomial with the guaranteed accuracy. The method uses computations with finite Toeplitz matrices and permits to obtain coefficients of both factorization factors simultaneously. Computation aspects of the algorithm are considered. An a priory estimate for the condition number of the used Toeplitz matrices is found. Formulas for computation of the Laurent coefficients with the given accuracy for functions that analytical and non-vanishing in an annular neighborhood of the unit circle are obtained. Stability of the factorization factors is studied. Upper bounds for the accuracy of the factorization factors are established. All estimates are effective. The proposed algorithm is implemented in Maple computer system as module "PolynomialFactorization". Numerical experiments with the module show a good agreement with the theoretical studies.

*Keywords: Wiener–Hopf factorization; polynomial factorization; Toeplitz matrices.*

## Introduction

Mathematical modelling of wave diffraction, problems of dynamic elasticity and fracture mechanics, and geophysical problems is reduced to so-called Wiener–Hopf factorization problem for matrix functions [1–3]. The factorization of matrix functions is also a powerful tool used in various areas of mathematics [4,5]. However for the matrix case there is no constrictive solution of the factorization problem in a general setting. Moreover, it is difficult to develop approximate methods for the problem since the matrix factorization is unstable. For this reason it is very important to find cases when the problem can be effectively or explicitly solved. The current state of this problem is presented in [6]. The first stage of an explicit method for solving of the factorization problem is the factorization of scalar functions. In turn, this problem can be reduced to the polynomial factorization [7].

Let $p(z) = p_0 + p_1 z + \cdots + p_\nu z^\nu$ be a complex polynomial of degree $\nu > 1$ and $p_0 \neq 0, p_\nu = 1$. Assume $p(z) \neq 0$ on the unit circle $\mathbb{T}$, hence $p(z) \neq 0$ on a closed circular annulus $K := \{z \in \mathbb{C} : r \leq |z| \leq R\}$ for some $0 < r < 1 < R < \infty$. By $\varkappa$ denote the number of zeros of $p(z)$ inside the unit circle. Let $\xi_j$, $j = 1, \ldots, \nu$, be the zeros of $p(z)$ and

$$0 < |\xi_1| \leq \ldots \leq |\xi_\varkappa| < r < 1 < R < |\xi_{\varkappa+1}| \leq \ldots \leq |\xi_\nu|.$$

Here the zeros are counted according to their multiplicity. Denote

$$p_1(z) := (z - \xi_1) \cdots (z - \xi_\varkappa),\ p_2(z) := (z - \xi_{\varkappa+1}) \cdots (z - \xi_\nu). \tag{1}$$

The representation

$$p(z) = p_-(z) z^\varkappa p_+(z), \quad |z| = 1, \tag{2}$$

where $p_-(z) = \frac{p_1(z)}{z^{\varkappa}}$, $p_+(z) = p_2(z)$, is called the Wiener–Hopf factorization, and

$$p(z) = p_1(z)p_2(z) \tag{3}$$

is the corresponding polynomial factorization of $p(z)$.

We consider coefficients of a polynomial $p(z)$ as initial data and coefficients of its factors as output data of the problem. The naive method of the polynomial factorization is to use formulas (1). However it is well known that roots of a polynomial are in general not well-condition functions of its coefficients (see, e.g., [8]), and coefficients of a polynomial are also not well-condition functions of its roots [9]. The latter means that, in general, we can not solve numerically the polynomial factorization problem by the naive way.

Nevertheless there exist numerical methods for solving of the problem. The basic works in this direction are cited in [10].

In this work we propose approach that is close to Algorithm 3 of D.A. Bini and A. Böttcher [10] but our algorithm permits to obtain coefficients of the factor $p_1(z)$, $p_2(z)$ simultaneously. In addition, we find effective upper bounds for the accuracy of the factorization factors. Note that our technique can be extended to the factorization problem for analytic functions. In this case we can obtain all coefficients of the polynomial factor and a required number of Taylor coefficients of the analytic factor.

## 1. Preliminaries

Throughout this paper, $\|x\|$ means the Hölder 1-norm $\|x\| = |x_1| + \cdots + |x_k|$, where $x = (x_1, \ldots, x_k)^T \in \mathbb{C}^k$. For a matrix $A \in \mathbb{C}^{\ell\times k}$ we always use the induced norm $\|A\| = \max_{1\le j\le k} \sum_{i=1}^{\ell} |A_{ij}|$.

Respectively, the norm of a polynomial $p(z) = p_0 + p_1 z + \cdots + p_\nu z^\nu$ is the norm of the vector $(p_0, p_1, \ldots, p_\nu)^T$. For $p(z)$ we will also apply the maximum norm $\|p\|_C = \max_{z\in\mathbb{T}} |p(z)|$ on the unit circle $\mathbb{T}$.

The norms $\|\cdot\|$ and $\|\cdot\|_C$ are equivalent. Clearly, $\|p\|_C \le \|p\|$. Since for $p(z)$ it is fulfilled the equality $\sum_{k=0}^{\nu} |p_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\varphi})|^2 d\varphi$, we have $\|p\| \le \sqrt{\nu+1}\,\|p\|_2 \le \sqrt{\nu+1}\,\|p\|_C$. Thus,

$$\|p\|_C \le \|p\| \le \sqrt{\nu+1}\,\|p\|_C. \tag{4}$$

In order to study stability of the factorization problem, we will need estimates for the norm of inverses of some Toeplitz matrices. Such estimates will be obtain in terms of $\|p_1\| \cdot \|p_2\|$, where $p_1(z)$, $p_2(z)$ are the factorization factors of $p(z)$. To get effective estimates, it will be required to estimate $\|p_1\|\,\|p_2\|$ via $\|p\|$.

Let $q(z) = q_1(z)q_2(z)$, where $q_1(z)$, $q_2(z)$ are arbitrary monic complex polynomials and $\nu = \deg q$. It is obvious that $\|q\| \le \|q_1\|\,\|q_2\|$. In the work [11], D.W. Boyd proved the following inequality $\|q_1\|_C\,\|q_2\|_C \le \delta^\nu \|q\|_C$, where $\delta = e^{2G/\pi} = 1,7916228120695934247\ldots$ and $G$ is Catalan's constant. The inequality is asymptoticaly sharp as $\nu \to \infty$.

Taking into account (4), in our case we obtain

$$\|p\| \le \|p_1\|\,\|p_2\| \le \delta^\nu \sqrt{(\varkappa+1)(\nu-\varkappa+1)}\,\|p\|. \tag{5}$$

However, the exponential factor $\delta^\nu$ can overestimate the upper bound. In some special cases we can obtain more precise estimates. For example, this can be done for a so-called spectral factorization of polynomials. In order to take into account a specific character of a given polynomial $p(z)$, we will use the inequalities

$$\|p\| \le \|p_1\|\,\|p_2\| \le \delta_0\|p\|, \tag{6}$$

instead of (5). Here $1 \le \delta_0 \le \delta^{\nu}\sqrt{(\varkappa+1)(\nu-\varkappa+1)}$.

## 2. Basic Tools

Let $M, N$ be integers, $M < N$, and $c_M^N = (c_M, c_{M+1}, \dots, c_N)$ a nonzero sequence of complex numbers. In this section we introduce notions of indices and essential polynomials for the sequence $c_M^N$. These notions were given in more general setting in the paper [12]. Here we will consider the scalar case only. The proofs of all statements of this section can be found in [12].

Let us form the family of Toeplitz matrices $T_k(c_M^N) = (c_{i-j})_{\substack{i=k,k+1,\dots,N,\\ j=0,1,\dots,k-M}}$, $M \le k \le N$, and study the sequence of the spaces $\ker T_k(c_M^N)$. Further it is more convenient to deal not with vectors $Q = (q_0, q_1, \dots, q_{k-M})^T \in \ker T_k$ but with their generating polynomials $Q(z) = q_0 + q_1\, z + \dots + q_{k-M}\, z^{k-M}$. We will use the spaces $\mathcal{N}_k$ of the generating polynomials instead of the spaces $\ker T_k$. The generating function $\sum_{j=M}^{N} c_k z^k$ of the sequence $c_M^N$ will be denoted by $c_M^N(z)$.

Let us introduce a linear functional $\sigma$ by the formula: $\sigma\{z^j\} = c_{-j}$, $-N \le j \le -M$. The functional is defined on the space of rational functions of the form $Q(z) = \sum_{j=-N}^{-M} q_j z^j$. Besides this algebraic definition of $\sigma$ we will use the following analytic definition

$$\sigma\{Q(z)\} = \frac{1}{2\pi i}\int\limits_{\Gamma} t^{-1} c_M^N(t) Q(t)\, dt. \tag{7}$$

Here $\Gamma$ is any closed contour around the point $z = 0$.

Denote by $\mathcal{N}_k$ $(M \le k \le N)$ the space of polynomials $Q(z)$ with the formal degree $k - M$ satisfying the orthogonality conditions:

$$\sigma\big\{z^{-i}Q(z)\big\} = 0,\ i = k, k+1, \dots, N. \tag{8}$$

It is easily seen that $\mathcal{N}_k$ is the space of generating polynomials of vectors in $\ker T_k$. For convenience, we put $\mathcal{N}_{M-1} = 0$ and denote by $\mathcal{N}_{N+1}$ the $(N-M+2)$-dimensional space of all polynomials with the formal degree $N - M + 1$. If necessary, the more detailed notation $\mathcal{N}_k(c_M^N)$ instead of $\mathcal{N}_k$ is used.

Let $d_k$ be the dimension of the space $\mathcal{N}_k$ and $\Delta_k = d_k - d_{k-1}$ $(M \le k \le N+1)$. The following inequalities are crucial for the further considerations: $0 = \Delta_M \le \Delta_{M+1} \le \ldots \le \Delta_N \le \Delta_{N+1} = 2$.

It follows from the inequalities that there exist integers $\mu_1 \le \mu_2$ such that

$$\begin{array}{lllllll} \Delta_M & = & \ldots & = & \Delta_{\mu_1} & = & 0, \\ \Delta_{\mu_1+1} & = & \ldots & = & \Delta_{\mu_2} & = & 1, \\ \Delta_{\mu_2+1} & = & \ldots & = & \Delta_{N+1} & = & 2. \end{array} \tag{9}$$

If the second row in these relations is absent, we assume $\mu_1 = \mu_2$.

**Definition 1.** *The integers $\mu_1$, $\mu_2$ will be called indices of the sequence $c_M^N$.*

It is easily seen that $\mathcal{N}_k$ and $z\mathcal{N}_k$ are subspaces of $\mathcal{N}_{k+1}$, $M-1 \le k \le N$. Let $h_{k+1}$ be the dimension of any complement $\mathcal{H}_{k+1}$ of the subspace $\mathcal{N}_k + z\mathcal{N}_k$ in the whole space $\mathcal{N}_{k+1}$.

From (9) we see that $h_{k+1} \ne 0$ iff $k = \mu_j$ $(j = 1, 2)$, $h_{k+1} = 1$ if $\mu_1 < \mu_2$, and $h_{k+1} = 2$ for $\mu_1 = \mu_2$. Therefore, $\mathcal{N}_{k+1} = \mathcal{N}_k + z\mathcal{N}_k$, for $k \ne \mu_j$, and $\mathcal{N}_{k+1} = \big(\mathcal{N}_k + z\mathcal{N}_k\big) \oplus \mathcal{H}_{k+1}$ for $k = \mu_j$

**Definition 2.** *Let $\mu_1 = \mu_2$. Any polynomials $Q_1(z)$, $Q_2(z)$ that form a basis for the two-dimensional space $\mathcal{N}_{\mu_1+1}$ are called the essential polynomials of the sequence $c_M^N$ corresponding to the index $\mu_1 = \mu_2$.*

*If $\mu_1 < \mu_2$, then any polynomial $Q_j(z)$ that is a basis for an one-dimensional complement $\mathcal{H}_{\mu_j+1}$ is said to be the essential polynomial of the sequence corresponding to the index $\mu_j$, $j = 1, 2$.*

## 3. Main Results

In this section we propose a method for solving the problem of the polynomial factorization in terms of indices and essential polynomials of some sequence. The proofs of the theorems of the section can be found in preprint [13].

Let $p(z) = p_0 + p_1 z + \cdots + p_\nu z^\nu$ be a polynomial of degree $\nu > 1$, $p_0 \ne 0$, $p(z) \ne 0$ on the unit circle $\mathbb{T}$, hence $p(z) \ne 0$ on a closed annulus $K := \{z \in \mathbb{C} : r \le |z| \le R\}$ for some $0 < r < 1 < R < \infty$. Let $\varkappa = \operatorname{ind}_{\mathbb{T}} p(z)$ be the number of zeros of the polynomial inside the unit circle. We can assume that $0 < \varkappa < \nu$, otherwise the factorizations are trivial. Put $n_0 = \max\{\varkappa,\ \nu - \varkappa\}$.

We will find the factorization of $p(z)$ in the form (3): $p(z) = p_1(z)p_2(z)$, where $\deg p_1 = \varkappa$, $\deg p_2 = \nu - \varkappa$, the polynomial $p_1(z)$ is monic, and all zeros of $p_1(z)$ (respectively $p_2(z)$) lie in the domain $\{z \in \mathbb{C} : |z| < r\}$ (respectively in $\{z \in \mathbb{C} : |z| > R\}$). This means that

$$p(z) = p_-(z)\, z^\varkappa p_+(z), \quad z \in \mathbb{T},$$

is the Wiener–Hopf factorization of $p(t)$ normalized by the condition $p_-(\infty) = 1$. Here $p_-(z) = \frac{p_1(z)}{z^\varkappa}$, $p_+(z) = p_2(z)$. Let $p^{-1}(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ be the Laurent series for analytic function $p^{-1}(z)$ in the annulus $K$,

$$c_k = \frac{1}{2\pi i} \int\limits_{|z|=\rho} t^{-k-1} p^{-1}(t)\, dt,\ k \in \mathbb{Z},\ r \le \rho \le R. \tag{10}$$

Form the sequence $c_{-n-\varkappa}^{n-\varkappa} = (c_{-n-\varkappa}, \dots, c_{-\varkappa}, \dots, c_{n-\varkappa})$ for the given $n \ge n_0$. It is easily to verify that the sequence is non-zero.

**Theorem 1.** *For any $n \ge n_0$ the integers $-\varkappa, -\varkappa$ are the indices, and $Q_1(z) = z^{n-\varkappa+1} p_1(z)$, $Q_2(z) = p_2(z)$ are essential polynomials of the sequence $c_{-n-\varkappa}^{n-\varkappa}$.*

From the theorem it follows that there exist the essential polynomials $Q_1(z), Q_2(z)$ of the sequence $c_{-n-\varkappa}^{n-\varkappa}$ such that the following additional properties are fulfilled:

1. $\deg Q_1(z) = n + 1, \quad Q_1(0) = 0, \quad Q_{1,n+1} = 1.$
2. $\deg Q_2(z) < n + 1, \quad Q_2(0) \ne 0, \quad \sigma\{z^\varkappa Q_2(z)\} = 1.$

Vice versa, if $Q_1(z), Q_2(z) \in \mathcal{N}_{-\varkappa+1}$ and satisfy conditions 1–2, then $Q_1(z), Q_2(z)$ are the essential polynomials of the sequence $c_{-n-\varkappa}^{n-\varkappa}$.

**Definition 3.** *Let $n \geq n_0$. Polynomials $Q_1(z), Q_2(z) \in \mathcal{N}_{-\varkappa+1}(c_{-n-\varkappa}^{n-\varkappa})$ satisfying conditions 1–2 will be called the factorization essential polynomials of $c_{-n-\varkappa}^{n-\varkappa}$.*

**Theorem 2.** *The factorization essential polynomials are uniquely determined by conditions 1–2. Let $n \geq n_0 + 1$, suppose that $R_1(z), R_2(z)$ are any essential polynomials of the sequence $c_{-n-\varkappa}^{n-\varkappa}$. Then the factorization essential polynomials of $c_{-n-\varkappa}^{n-\varkappa}$ can be found by the formulas*

$$Q_1(z) = \frac{1}{\sigma_1} \begin{vmatrix} R_1(z) & R_2(z) \\ R_{1,0} & R_{2,0} \end{vmatrix}, \quad Q_2(z) = \frac{1}{\sigma_0} \begin{vmatrix} R_1(z) & R_2(z) \\ R_{1,n+1} & R_{2,n+1} \end{vmatrix}. \tag{11}$$

*Here $R_{j,0} = R_j(0)$, $R_{j,n+1}$ is the coefficient of $z^{n+1}$ in the polynomial $R_j(z)$, $j = 1, 2$, and*

$$\sigma_1 = \begin{vmatrix} R_{1,0} & R_{2,0} \\ R_{1,n+1} & R_{2,n+1} \end{vmatrix} \neq 0, \quad \sigma_0 = \sigma \left\{ z^{\varkappa} \begin{vmatrix} R_1(z) & R_2(z) \\ R_{1,n+1} & R_{2,n+1} \end{vmatrix} \right\} \neq 0.$$

Now we can construct the Wiener–Hopf factorization of a polynomial with the help of the factorization essential polynomials.

**Theorem 3.** *Let $n \geq n_0$ and let $Q_1(z), Q_2(z)$ be the factorization essential polynomials of the sequence $c_{-n-\varkappa}^{n-\varkappa}$. Then the Wiener–Hopf factorization of the polynomial $p(z)$ can be constructed by the formula $p(z) = p_-(z) z^{\varkappa} p_+(z)$, where*

$$p_-(z) = z^{-n-1} Q_1(z), \quad p_+(z) = Q_2(z). \tag{12}$$

The following theorem about explicit formulas for the factors of the Wiener–Hopf factorization of a polynomial $p(z)$ is the main result of the section.

**Theorem 4.** *The matrices $T_{-\varkappa}\left(c_{-n-\varkappa}^{n-\varkappa}\right)$ are invertible for all $n \geq n_0$.*

*Let $n \geq n_0 + 1$. Denote by $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $\beta = (\beta_0, \ldots, \beta_n)^T$ the solutions of the systems*

$$T_{-\varkappa}\left(c_{-n-\varkappa+1}^{n-\varkappa-1}\right) \alpha = -(c_{-n-\varkappa}^{-\varkappa-1})^T, \quad T_{-\varkappa}\left(c_{-n-\varkappa}^{n-\varkappa}\right) \beta = e_1, \tag{13}$$

*respectively. Here $e_1 = (1, 0, \ldots, 0)^T$.*

*Then $\alpha_1 = \ldots = \alpha_{n-\varkappa} = 0$, $\beta_0 \neq 0$, $\beta_{\nu-\varkappa+1} = \ldots = \beta_n = 0$, and the factors from the Wiener–Hopf factorization of $p(z)$ are found by the formulas*

$$p_-(z) = z^{-\varkappa}(\alpha_{n-\varkappa+1} + \cdots + \alpha_n z^{\varkappa-1} + z^{\varkappa}), \quad p_+(z) = \beta_0 + \beta_1 z + \cdots + \beta_{\nu-\varkappa} z^{\nu-\varkappa}.$$

## 4. Some Computational Aspects of the Factorization Problem

In this section we consider some computation problems that arise in a numerical implementation of the results of Section 3.Our aim is to obtain the factors $p_1(z)$, $p_2(z)$ with the guaranteed accuracy. The proofs of the statements of the section can be found in preprint [13].

### 4.1. An a Priori Estimate of the Condition Number for the Factorization Problem

Theorem 4 shows that solving of the factorization problem is equivalent to solving of linear systems with the invertible matrix $T_{-\varkappa}(c_{-n-\varkappa}^{n-\varkappa})$, $n \geq n_0 := \max\{\varkappa, \nu - \varkappa\}$.

Here we obtain an upper bound for the condition number $k(T_{-\varkappa}(c_{-n-\varkappa}^{n-\varkappa})) = \|T_{-\varkappa}(c_{-n-\varkappa}^{n-\varkappa})\| \, \|T_{-\varkappa}^{-1}(c_{-n-\varkappa}^{n-\varkappa})\|$ in terms of the given polynomial $p(z)$.

Denote $m_K = \min_{z\in K} |p(z)|$, $m_1 = \min_{|z|=1} |p(z)|$, $\rho = \max\{r, 1/R\}$. Then the following estimate of the condition number is fulfilled.

**Theorem 5.** *For $n \geq \nu$*

$$k(T_{-\varkappa}(c_{-n-\varkappa}^{n-\varkappa})) \leq \min\left\{\frac{\delta_0}{m_K}\,\frac{1+\rho}{(1-\rho)}\|p\|,\ \frac{(2n+1)\delta_0}{m_1}\,\|p\|\right\}. \tag{14}$$

### 4.2. Computation of the Laurent Coefficients of Analytic Functions

To realize the factorization method proposed in Section 3 we must calculate the Laurent coefficients $c_{-n-\varkappa}, \ldots, c_{-\varkappa}, \ldots, c_{n-\varkappa}$, of the function $p^{-1}(z)$ for $n \geq n_0 = \max\{\varkappa, \nu - \varkappa\}$.

In general, the coefficients can be found only approximately. In order to do this, we will apply the method suggested by D.A. Bini and A. Böttcher (see Theorem 3.3 in [10]). For future applications we will consider more general situation than in this work. Moreover, our proof differs from the proof in the above mentioned paper.

Let $f(z)$ be a function that analytic in the annulus $K = \{z \in \mathbb{C} : r \leq |z| \leq R\}$, $0 < r < 1 < R < \infty$. By $f_k$ denote the Laurent coefficients of $f(z)$: $f_k = \frac{1}{2\pi i} \int\limits_{|t|=\rho} t^{-k-1} f(t)\, dt$, $r \leq \rho \leq R$. For $\ell, k \in \mathbb{Z}$, $\ell \geq 2$, define $\widetilde{f}_k(\ell) = \frac{1}{\ell}\sum_{j=0}^{\ell-1} \frac{f(\omega_j)}{\omega_j^k}$, where $\omega_j = e^{\frac{2\pi i}{\ell} j}$ are the zeros of the polynomial $z^\ell - 1$.

**Theorem 6.** *Let $M_K = \max\limits_{z\in K} |f(z)|$, $\rho = \max\{r, 1/R\}$, and $\ell$ be an even positive integer. Then*

$$|f_k - \widetilde{f}_k(\ell)| < \frac{2M_K}{(1-\rho^\ell)}\rho^{\ell/2} \tag{15}$$

*for $k = -\ell/2, \ldots, 0, \ldots, \ell/2$.*

By the theorem, in order to compute every element of the sequence $f_M, f_{M+1}, \ldots, f_N$ with the given accuracy, we have to select an appropriate number $\ell$.

### 4.3. Stability of the Factors $p_1(z)$, $p_2(z)$

Now we study the behavior of the factors $p_1$, $p_2$ under small perturbations of $p(z)$. Let $m_1 = \min\limits_{|z|=1} |p(z)|$. It is easily seen that $\widetilde{p}(z) \neq 0$ on $\mathbb{T}$ and $\operatorname{ind}_{\mathbb{T}} p(z) = \operatorname{ind}_{\mathbb{T}} \widetilde{p}(z)$ if $\|p - \widetilde{p}\| < m_1$. Let $\widetilde{p}(z) = \widetilde{p}_1(z)\widetilde{p}_2(z)$ be the factorization of $\widetilde{p}(z)$. By $\widetilde{c}_j$ we denote the Laurent coefficients of $\widetilde{p}^{-1}(z)$. Estimate $\|p_1 - \widetilde{p}_1\|$, $\|p_2 - \widetilde{p}_2\|$ via $\|p - \widetilde{p}\|$.

**Theorem 7.** *Let $n \geq n_0 + 1$. If $\|p - \widetilde{p}\,\| \leq \min\left\{\frac{m_1}{2}, \frac{m_1^2}{4(2n+1)\delta_0\|p\|}\right\}$, then*

$$\|p_1 - \widetilde{p}_1\| < \frac{4(2n+1)\delta_0\|p\|}{m_1^2}\left[\frac{\delta_0\|p\|}{m_K}\frac{1+\rho}{1-\rho} + 1\right]\|p - \widetilde{p}\,\|, \tag{16}$$

*and*

$$\|p_2 - \widetilde{p}_2\| < \frac{4(2n+1)\delta_0^2\|p\|^2}{m_1^2}\|p - \widetilde{p}\,\|. \tag{17}$$

Now we consider perturbations of the polynomials $p_1(z)$, $p_2(z)$ caused by the approximation of the sequence $c_{-n-\varkappa}^{n-\varkappa}$ by $\widetilde{c}_{-n-\varkappa}^{n-\varkappa}$, where $\widetilde{c}_k = \frac{1}{\ell}\sum_{j=0}^{\ell-1}\frac{1}{\widetilde{p}(\omega_j)\omega_j^k}$. Let $\widetilde{p}_1(z)$, $\widetilde{p}_2(z)$ be polynomials that define by Eq. (13) for the sequence $\widetilde{c}_{-n-\varkappa}^{n-\varkappa}$.

**Theorem 8.** *Let $n \geq n_0 + 1$, $\ell$ is an even integer such that $\ell \geq 2(n + \varkappa)$, and*

$$\frac{\rho^{\ell/2}}{1-\rho^\ell} < \frac{m_K}{2(4n+2)\delta_0\|p\|}. \tag{18}$$

*Then*

$$\|p_1 - \widetilde{p}_1\| < \frac{2(4n-2)\delta_0\|p\|}{m_K}\left[\frac{2\delta_0(1+\rho)\|p\|}{(1-\rho)m_K} + 1\right]\frac{\rho^{\ell/2}}{1-\rho^\ell},$$

$$\|p_2 - \widetilde{p}_2\| < \frac{2(4n+2)\delta_0^2\|p\|^2}{m_K}\frac{\rho^{\ell/2}}{1-\rho^\ell}.$$

From the theorem it is easy to find the estimate of $\ell$ in order to obtain the factors $p_1(z)$, $p_2(z)$ with the given accuracy $\varepsilon$.

**Corollary 1.** *Let $0 < \varepsilon < 1$ and*

$$\alpha = \varepsilon\frac{m_K}{2\delta_0\|p\|}\min\left\{(4n-2)\left(1 + \frac{\delta_0\|p\|(1+\rho)}{m_K(1-\rho)}\right), (4n+2)\delta_0\|p\|\right\}.$$

*If $\ell$ is an even integer such that*

$$\ell > 2\max\left\{n + \varkappa, \frac{\log\left(\sqrt{1 + \frac{1}{4\alpha^2}} + \frac{1}{2\alpha}\right)}{|\log\rho|}\right\}, \tag{19}$$

*then $\|p_1 - \widetilde{p}_1\| < \varepsilon$, $\|p_2 - \widetilde{p}_2\| < \varepsilon$.*

# 5. Algorithm and Its Implementation

## 5.1. Algorithm

The above results can be summarized in the form of the following algorithm.

**Algorithm.** Polynomial factorization

INPUT. The polynomial $p$, the parameter par giving a variant of estimate of $\|p_1\| \cdot \|p_2\|$ via $\|p\|$ (par $= 1, 2, 3$), the integer $n > \deg p$, the accuracy $\Delta = 10^{-q}$ for $p$: $\|p - \widetilde{p}\| < \Delta$.
OUTPUT. The polynomials $p_1$, $p_2$ obtained with the guaranteed accuracy $\varepsilon$, $\varepsilon$.

COMPUTATION.

1. Compute the index $\varkappa$ of $p$.
2. Compute the radii $r, R$ of the circular annulus $K$ and $\rho = \max\{r, 1/R\}$, compute $m_1 = \min_{|z|=1} |p(z)|$, $m_K = \min_{z \in K} |p(z)|$.
3. Find accuracy $\varepsilon_1$, $\varepsilon_2$ for $p_1$, $p_2$ by formulas (16), (17). Compute the theoretically guaranteed accuracy $\varepsilon := \max\{\varepsilon_1, \varepsilon_2\}$. Define $d$ such that $\varepsilon < 10^{-d}$.
4. To compensate the loss of accuracy under calculations with the Toeplitz matrix $T_{-\varkappa}(c_{-n-\varkappa}^{n-\varkappa})$ we must improve accuracy of Laurent coefficients $c_k$. To do this we estimate the condition number $k$ of the matrix by formula (14), find $\widetilde{d}$ such that $k \le 10^{\tilde{d}}$, and put $\widetilde{\varepsilon} := 10^{-d-\tilde{d}}$.
5. Find an even integer $\ell$ satisfying inequality (19) (in the inequality put $\varepsilon = \widetilde{\varepsilon}$).
6. Form the sequence $\widetilde{c}_{-n-\varkappa}^{\,n-\varkappa}$.
7. Form the Toeplitz matrix $T_{-\varkappa+1}(\widetilde{c}_{-n-\varkappa}^{\,n-\varkappa})$ and find a basis $\{R_1,\, R_2\}$ of its kernel. The last operation can be done with the help of SVD.
8. Find the factorization essential polynomials $Q_1(z) = \alpha_1 z + \dots + \alpha_n z^n + z^{n+1}$, $Q_2(z) = \beta_0 + \beta_1 z + \dots + \beta_n z^n$ by (11).
9. Verify that the absolute values of the coefficients $\alpha_1, \dots, \alpha_{n-\varkappa}$, $\beta_{\nu-\varkappa+1}, \dots, \beta_n$ are less than $\varepsilon$ and delete these coefficients (see Theorem 4).
10. $p_1(z) := z^{-n+\varkappa-1} Q_1(z)$, $p_2(z) := Q_2(z)$, $\varepsilon$.
11. end

### 5.2. Implementation in Maple

The algorithm was implemented in Maple as the module "PolynomialFactorization". The module can work in Maple 17. To access "PolynomialFactorization", use the commands

```
> read("PolynomialFactorization.mpl");
> with(PolynomialFactorization);
```

Then it is necessary to set the value of **Digits** taking into account initial data. If the user needs the results of intermediate calculations, then to set

```
> PrintOn:= true;
```

The basic routines of "PolynomialFactorization" are **indpoly, annulusn (annulus), condnpoly, getellpoly, LaurentCoeff, factesspoly, SolverPF**.

- **indpoly** returns the index $\varkappa$ of the polynomial $p$ found by formula (12.6) from [14].
- **annulusn** returns $\rho$, $m_K$.

- **condnpoly** returns the estimate of the condition number $k$ by formula (14).
- **getellpoly** returns the integer $\ell$ required for calculation of the Laurent coefficients with the given accuracy $\widetilde{\varepsilon}$.
- **LaurentCoeff** returns the sequence $c_M^N$ of the Laurent coefficients of the function $1/p$.
- **factesspoly** returns the factorization essential polynomials.
- **SolverPF** returns the factors $p_1$, $p_2$, and the guaranteed accuracy $\varepsilon$ of calculation.

The polynomial factorization is realized by routine **SolverPF**. The following input data are passed to **SolverPF**:

- a polynomial $p$ in a variable $z$.
- the parameter par taking the values $1, 2, 3$. A value of par gives a variant of estimate of $\|p_1\| \cdot \|p_2\|$ via $\|p\|$. If $p = \sum_{j=0}^{2m} p_j z^j$ be a real polynomial of degree $2m$ such that $p_{2m-j} = p_j$ for $j = 0, \ldots, m$, $p_0 = 1$, and all roots of $p(z)$ have negative real parts, then par $= 1$. If $p = \sum_{j=0}^{2m} p_j z^j$ be a complex polynomial of degree $2m$ such that $p_{2m-j} = p_j$ for $j = 0, \ldots, m$, $p_0 = 1$, then par $= 2$. If $p$ is a polynomial of a general kind, then par $= 3$.
- an integer $n > \deg p$, where $2n + 1$ is the length of the used sequence $c_{-n-\varkappa}^{n-\varkappa}$.
- the accuracy $\Delta = 10^{-q}$ of calculation $p$.

Here is the example to call the program.

```
> Digits:= 15;
> p:= 1+3iz/2+z^2;
> p1, p2, epsilon := SolverPF(p, 2, 3, 10^(-10)):
                "Calculation in progress ...."
                     "End of calculation"
> p1;
          1.00000000000000*z-0.500000000000004*I

> p2;
          1.00000000000001*z+1.99999999999999*I
> epsilon;
          0.00000755534207942884
```

## 6. Numerical Examples

In the following examples we use module "PolynomialFactorization". All computations were performed on a desktop.

The polynomial $p(z)$ in Example 1 satisfies the conditions of Proposition 2.2 from [13], and its Wiener – Hopf factorization is actually the spectral factorization.

**Example 1.** Let $p(z) = (z+1/2)(z+1/3)\cdots(z+1/12)(z+2)(z+3)\cdots(z+12)$. Taking into account the values of the coefficients of $p(z)$, we choose the precision `Digits` := 20. Assume that the accuracy of the input data $\Delta$ is equal to $10^{-15}$.

We have $\nu = 22$, $\varkappa = 11$, $\|p\| = 20237817600$. Computations show that $\rho := 0,51$, $m_1 = 3,326340 \times 10^6$, $m_K = 30,448076$. Put $n = \nu + 1 = 23$. Here par := 1 and $\delta_0 = 1$. Calculation of the theoretically guaranteed accuracy $\varepsilon$ gives the following result $\varepsilon = 0,695883 \times 10^{-5}$. By formula (14), we obtain the following estimate $k(T_{-\varkappa}(c^{n-\varkappa}_{-n-\varkappa})) \leq 2,859480 \times 10^5$. It follows from this that $\widetilde{\varepsilon} = 10^{-22}$ and we get $\ell = 136$.

In this example the exact output is known

$$p_1(z) = (z+1/2)(z+1/3)\cdots(z+1/12), \quad p_2(z) = (z+2)(z+3)\cdots(z+12).$$

Table 1 shows the results of approximate computations of the factors $\tilde{p}_1(z)$, $\tilde{p}_2(z)$. It contains coefficients $\tilde{p}^1_k$, $\tilde{p}^2_k$, absolute errors $|\tilde{p}^1_k - p^1_k|$, $|\tilde{p}^2_k - p^2_k|$ for the coefficients $p^1_k$, $p^2_k$, and $\|\tilde{p}_1 - p_1\|$, $\|\tilde{p}_2 - p_2\|$. For $\tilde{p}^1_k$, $\tilde{p}^2_k$ the number of decimal places obtained accurately is shown.

**Table 1**

Coefficients $\tilde{p}^1_k$, $\tilde{p}^2_k$

| $k$ | $\tilde{p}^1_k$ | $\lvert\tilde{p}^1_k - p^1_k\rvert$ | $\tilde{p}^2_k$ | $\lvert\tilde{p}^2_k - p^2_k\rvert$ |
|---|---|---|---|---|
| 0 | 0 | 2,087675e-9 | 479001600,00000 | 1,04000e-9 |
| 1 | 0 | 1,60751e-7 | 1007441280,00000 | 1,26000e-8 |
| 2 | 0 | 0,55114e-5 | 924118272,00000 | 3,78100e-8 |
| 3 | 0,00011 | 1,97436e-18 | 489896616,00000 | 5,78000e-8 |
| 4 | 0,00145 | 4,98731e-17 | 167310220,00000 | 5,46800e-8 |
| 5 | 0,01300 | 9,54140e-17 | 38759930,00000 | 3,62700e-8 |
| 6 | 0,08091 | 1,20571e-16 | 6230301,00000 | 2,20830e-8 |
| 7 | 0,34928 | 1,18000e-16 | 696333,00000 | 1,81875e-8 |
| 8 | 1,02274 | 8,87000e-17 | 53130,00000 | 1,72958e-8 |
| 9 | 1,92925 | 4,76000e-17 | 2640,00000 | 1,24386e-8 |
| 10 | 2,10321 | 1,40000e-17 | 77,00000 | 2,80480e-9 |
| 11 | 1,00000 | 0 | 0,999999 | 9,23655e-9 |
| $\|\tilde{p}_1 - p_1\|$ | | 0,56743e-5 | | |
| $\|\tilde{p}_2 - p_2\|$ | | | | 2,82246e-7 |

Thus $\|\tilde{p}_1 - p_1\| = 0,56743 \times 10^{-5} < 0,695884 \times 10^{-5} = \varepsilon$, and $\|\tilde{p}_2 - p_2\| = 2,82246 \times 10^{-7} < 0,695884 \times 10^{-5} = \varepsilon$. We obtain $p_1(z)$, $p_2(z)$ with the desired accuracy.

The following example was taken from [10]. Since $p(z)$ has real coefficients $p_j$ and $p_{\nu-j} = p_j$, then the factorization of $p(z)$ is also the spectral factorization.

**Example 2.** Let $p(z) = \sum_{i=0}^{10} z^i + 4z^5$, `Digits` := 20, $\Delta = 10^{-12}$. Now $\rho = 0,83$, $m_1 = 1,542464$, $m_K = 0,062855$.

In this example $\nu = 10$, $\varkappa = 5$, $\|p\| = 15$, $n = \nu+1 = 11$, par := 2, $\delta_0 = \kappa+1 = 6$. For the accuracy $\varepsilon$ we obtain $\varepsilon = 0,536458 \times 10^{-4}$. From formula (14) it follows the following estimate $k(T_{-\varkappa}(c^{n-\varkappa}_{-n-\varkappa})) \leq 1342,008991$. This yields $\widetilde{\varepsilon} = 10^{-17}$ and $\ell = 418$.

**Table 2**

Coefficients $\tilde{p}_k^1$, $\tilde{p}_k^2$

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\tilde{p}_k^1$ | 0,23193 | 0,20715 | 0,17674 | 0,14253 | 0,10685 | 1,00000 |
| $\tilde{p}_k^2$ | 4,31154 | 0,46071 | 0,61452 | 0,76203 | 0,89314 | 1,00000 |

The computed coefficients of the factors $\tilde{p}_1(z)$, $\tilde{p}_2(z)$ are given by Table 2. We only indicate 5 decimal places here.

In order to verify the computation correctness of $p_2(z)$, we can use the following relation between the factors $p_1(z)$ and $p_2(z)$ in the spectral factorization: $p_2(z) = z^{\varkappa} p_1(1/z)/p_1(0)$. For our example we have $\|\tilde{p}_2 - z^{\varkappa}\tilde{p}_1(1/z)/\tilde{p}_1(0)\| = 7,36 \times 10^{-18}$. Moreover, the residual error is $\|\tilde{p}_1\tilde{p}_2 - p\| = 8,1 \times 10^{-18}$.

In the next example the random polynomial $p(z)$ was generated with the help of package `Random Tools`.

**Example 3.** Let

$$p = z^{11} - \frac{17}{30}z^{10} + \frac{13}{10}z^9 + \left(\frac{223}{60} + \frac{848}{135}i\right)z^8 + \left(-\frac{28}{15} + \frac{514}{135}i\right)z^7 + \\ + \left(-\frac{43}{60} + \frac{106}{135}i\right)z^6 + \left(\frac{43}{60} + \frac{764}{135}i\right)z^5 + \left(-\frac{31}{6} + \frac{68}{135}i\right)z^4 + \left(\frac{7}{3} - \frac{2}{3}i\right)z^3 + \\ + \left(-1 + \frac{814}{135}i\right)z^2 + \left(\frac{39}{10} + \frac{58}{15}i\right)z + \left(-\frac{61}{60} + \frac{16}{9}i\right),$$

`Digits` := 20, $\Delta = 10^{-18}$. Calculations show that $\rho = 0,943396$, $m_1 = 2,293009$, $m_K = 0,66281$.

**Table 3**

Coefficients $\tilde{p}_k^1$, $\tilde{p}_k^2$

| $k$ | $\tilde{p}_k^1$ | $\tilde{p}_k^2$ |
|---|---|---|
| 0 | $-0,099841 - 0,150475i$ | $-5,090491 - 10,133912i$ |
| 1 | $-0,236722 + 0,118527i$ | $-14,129949 + 0,552043i$ |
| 2 | $-0,385402 - 0,732498i$ | $-4,543939 + 4,838437i$ |
| 3 | $1,00000$ | $-7,958489 + 1,840704i$ |
| 4 | | $-5,515909 + 9,645327i$ |
| 5 | | $4,196252 + 7,320240i$ |
| 6 | | $0,930308 + 0,031004i$ |
| 7 | | $-0,181264 + 0,732498i$ |
| 8 | | $1,000000$ |

For the polynomial we have $\nu = 11$, $\varkappa = 3$, $n = \nu + 1 = 12$, $\|p\| = 42,442968$. Since $p(z)$ is a polynomial of general type, par := 3, $\delta_0 = 3663,225630$ is found by the formula $\delta_0 = \delta^{\nu}\sqrt{(\varkappa+1)(\nu-\varkappa+1)}$. By this reasons, we are forced to use more accurate input data. Then the guaranteed accuracy in the output is $\varepsilon = 0,254667 \times 10^{-4}$.

From formulas (14) we obtain the estimate $k(T_{-\varkappa}(c_{-n-\varkappa}^{n-\varkappa})) \leq 1342,00899$. Hence $\widetilde{\varepsilon} = 10^{-26}$ and $\ell = 2096$.

The computed coefficients of the factors $\tilde{p}_1(z)$, $\tilde{p}_2(z)$ are given by Table 3.The residual error is $\|\tilde{p}_1\tilde{p}_2 - p\| = 4,354070 \times 10^{-7}$.

Let $\hat{p}_1(z)$, $\hat{p}_2(z)$ be the factorization factors of $p(z)$ obtained by the naive method (via the roots of $p(z)$). Then $\|\tilde{p}_1 - \hat{p}_1\| = 1.3 \times 10^{-10}$, $\|\tilde{p}_2 - \hat{p}_2\| = 5,239393 \times 10^{-8}$.

**Acknowledgement.** *The program module "PolynomialFactorization" was designed in collaboration with Vl.M. Adukov. The author is grateful to him for his invaluable help in programming.*

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*Received July 20, 2018*

УДК 519.688 DOI: 10.14529/mmp180408

# АЛГОРИТМ ПОЛИНОМИАЛЬНОЙ ФАКТОРИЗАЦИИ И ЕГО ИМПЛЕМЕНТАЦИЯ В MAPLE

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В работе предложен алгоритм факторизации Винера – Хопфа скалярных многочленов. Алгоритм, основанный на понятиях индексов и существенных многочленов, позволяет найти факторизационные множители многочлена с гарантированной точностью. Метод использует вычисления с конечными теплицевыми матрицами и дает возможность получить коэффициенты обоих факторизационных факторов одновременно. Рассмотрены вычислительные аспекты алгоритма. Найдена априорная оценка числа обусловленности используемой теплицевой матрицы. Получены формулы для вычисления лорановских коэффициентов с заданной точностью для функций аналитических и не обращающихся в нуль в кольцевой окрестности единичной окружности. Изучена устойчивость факторизационных множителей. Установлены верхние границы точности вычисления факторизационных множителей. Все оценки являются эффективными. Предложенный алгоритм был реализован в компьютерной системе Maple в виде модуля «PolynomialFactorization». Численные эксперименты с модулем показали хорошее согласие с теоретическим исследованием.

*Ключевые слова: факторизация Винера – Хопфа; полиномиальная факторизация; теплицевы матрицы.*

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*Поступила в редакцию 20 июля 2018 г.*