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CONSISTENCY AND LYAPUNOV STABILITY OF LINEAR SINGULAR TIME DELAY SYSTEMS: A GEOMETRIC APPROACH

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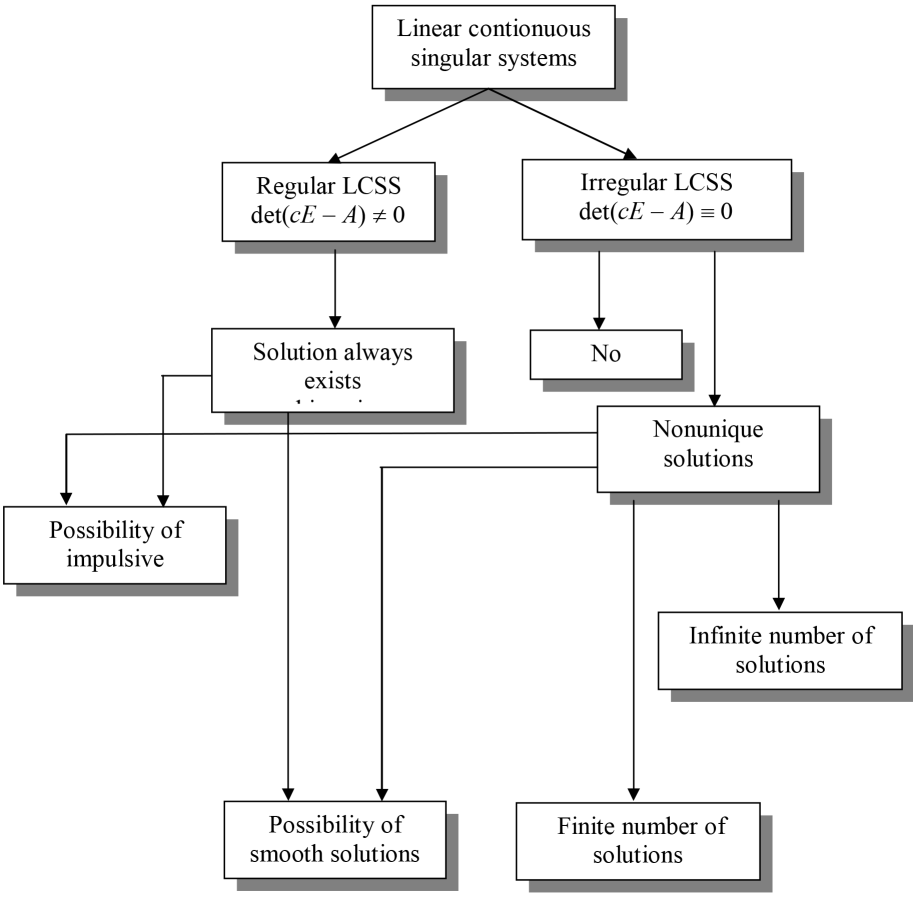
When we consider the control design of practical systems (chemical engineering systems, lossless transmission lines, large-scale electric network control, aircraft attitude control, flexible arm control of robots, etc.), time-delay often appears in many situations. Singular time delayed systems are the dynamic systems described by a mixture of algebraic and differential equations with retarded argument. This paper investigates the geometric description of initial conditions that generate smooth solutions to such problems and the construction of the Lyapunov stability theory to bound the rates of decay of such solutions. The new delay dependent conditions for asymptotic stability for the class of systems under consideration were derived. Moreover, the result is expressed in terms of only systems matrices that naturally occur in the model, therefore avoiding the need to introduce algebraic transformations into the statement of the main theorem.

Keywords: singular time delayed systems; stability in the sense of Lyapunov; consistent initial conditions.

Dedicated to Professor Viktor Chistyakov
on the occasion of his 70th birthday.

Introduction

It is well-known that in some systems we should consider their dynamics and statics simultaneously. Singular systems (also referred to degenerate, descriptor, generalized, differential-algebraic or semi-state systems) are systems which dynamic is governed by the complexity of algebraic and differential equations. Recently, many researchers have paid much attention to singular systems so they have accomplished numerous interesting conclusions. The complex nature of singular systems generates many difficulties in their analytical and numerical solution, particularly, when we deal with control problems. Recently, time delay singular systems have been one of the major research fields of the control theory. During the past three decades singular systems have attracted much attention due to the comprehensive applications in economics (e.g. the Leontief dynamic model), in electrical applications using the theory described in [1], in mechanical models [2], etc. Singular systems in control theory were initially discussed in [3] and [4]. Such systems are represented by the combination of the differential and algebraic equations using the state-space formalism. Due to the existence of the algebraic equations (constraints to the system), the investigation of singular systems is more complicated than the study of regular systems. The survey of updated results for singular systems and the broad bibliography can be found in [1, 5–11] and in the two special issues *Circuits, Systems and Signal Procesing* [12, 13].



Possibilities of existence, uniqueness and the number of linear continuous singular solutions

A specific nature of singular systems is well documented in the figure. Models in this form have some important advantages in comparison with the models in the normal form, e.g. when $E = I$. These models preserve the sparsity of the system matrices (many entries of system matrices are equal to zero). There is a tight connection between the system physical variables and the variables in the model. The structure of the physical system is well reflected in the model. These equations are easily derived and it is not required to eliminate the unwanted (redundant) variables, as there is no need for the formulation of the state variables. By now, the scientific community have comprehensively investigated time delay systems. The engineering practice required some practical solutions, including stability investigations in various technical systems, such as the electric, pneumatic and hydro-electrical complex systems, processes in chemical industries, complex transmission systems, etc. Time delays present in the system state variables or in the control signals can be the cause of undesirable system performances including inadequate transient response or instability. Consequently, stability analysis of such systems became one of the major topics in many research studies. Generally speaking, the existence of time lag and its corresponding components makes the investigations more demanding in adopting the adequate mathematical tools.

Investigation of the systems with time delay includes mainly two approaches. The first one implies finding the condition for the system stability that does not incorporate any knowledge related to the time delay. The second method utilizes the system delay (lag, latency) that is incorporated in the conditions for system stability. The first group of mathematical equations is usually referred as the delay-independent criteria, and it generally provides algebraic conditions that can be applied in calculations without additional complications. When we consider the control design of practical systems (chemical engineering systems, lossless transmission lines, large-scale electric network control, aircraft attitude control, flexible arm control of robots, etc.), time-delay often appears in many situations. When time delay is small, it can be ignored. If it is large, however, it may cause instability in the system. We should emphasize that many systems have the phenomena of time delay and singularity simultaneously. We call such systems *singular time delay differential systems.* Such systems have many special characteristics, and their investigation is not a trivial task. In this paper we will discuss the recent advances in this area and present some new results.

1. Nomenclature and Preliminaries

Many research papers have addressed the Lyapunov stability of particular classes of linear singular time delay systems. They usually employ the LMI approach [14–16].

In this paper, our results are based on the second Lyapunov method and the geometric approach. In that sense, these results can be treated as the further extension of the papers [17–22] providing contributions in the form of the weak Lyapunov algebraic matrix equation with some additional constraints. We suggest a new approach to the stability of singular time delay systems. The results are directly expressed in terms of matrices E, A_0 and A_1 naturally occurring in the system model. In this approach there is no need to introduce any canonical form in the statement of theorems. The geometric consistency theory provides a natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes the construction of the Lyapunov stability theory possible even for the linear continuous singular time-delay systems. Moreover, the attractive property is equivalent to the existence of symmetric positive definite solutions of a weak form to the Lyapunov matrix equation, incorporating conditions which refer to the boundedness of solutions.

The following denotations are used in this study: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{C}^n$ is a complex vector space and $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $(n \times m)$. Superscript $\top$ stands for the matrix transposition. $X > 0$ means that X is a real symmetric and positive definite. I stands for the identity matrix. If dimensions of matrices are not explicitly stated, they are assumed to be appropriate for algebraic operations.

Consider the generalized equation for the time delay singular control systems in its differential form:

$$\begin{gathered} E(t)\,\dot{\mathbf{x}}(t) = \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{x}(t-\tau), \mathbf{u}(t)\right),\ t \geq 0, \\ \mathbf{x}(t) = \phi(t),\ -\tau \leq t \leq 0, \end{gathered} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control vector, $E(t) \in \mathbb{R}^{n\times n}$ is a singular matrix, $\phi \in C([-\tau, 0],\ \mathbb{R}^n)$ is an admissible initial state functional, $C([-\tau, 0],\ \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with topology of uniform convergence. The state vector function satisfies:

$$\mathbf{f}(\cdot) : \mathfrak{I} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \tag{2}$$

and it is assumed to be smooth to guarantee the existence and uniqueness of solutions over an infinite time interval.

The same vector function has the continuous dependence of the solutions denoted by $\mathbf{x}(t, t_0, \mathbf{x}_0)$ with respect to t and the initial data. $\mathbb{R}^n$ is the state space of system (1) and $\|\cdot\|$ is the Euclidean norm. $V : \mathfrak{I} \times \mathbb{R}^n \to \mathbb{R}^n$ is the tentative aggregate function, so that $V(t, \mathbf{x}(t))$ is limited and for which $\|\mathbf{x}(t)\|$ is limited as well.

The Eulerian derivative of $V(t, \mathbf{x}(t))$ along with the trajectory of (1) is defined as:

$$\dot{V}(t, \mathbf{x}(t)) = \frac{\partial V(t, \mathbf{x}(t))}{\partial t} + [\text{grad}\, V(t, \mathbf{x}(t))]^T \Psi \mathbf{f}(\cdot), \tag{3}$$

where matrix Ψ is the solution of following matrix equation [23]:

$$[\text{grad}\, V(t, \mathbf{x}(t))]^T = [\text{grad}\, V(t, \mathbf{x}(t))]^T \Psi E. \tag{4}$$

In this study we focus our attention on the linear continuous time delay singular systems:

$$E\dot{\mathbf{x}}(t) = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t - \tau), \tag{5}$$

where the compatible vector valued function of the initial conditions is known and has the form

$$\mathbf{x}(t) = \phi(t), \quad -\tau \le t \le 0, \tag{6}$$

matrices A_0 and A_1 are constant matrices of adequate dimensions. It is assumed that $\operatorname{rank} E = r < n$.

In the further analysis we consider the case when the subspace of consistent initial conditions for a singular time delay system coincides with that of the singular non-delay system.

Remark 1. The singularity of matrix E ensures that solutions to (5) exist only for special values of $\phi(t) \in \mathcal{W}^*_{cont}\ \forall t \in [-\tau, 0]$.

It was shown in [24] for the singular system (5) without delay that the subspace of consistent initial conditions $\mathcal{W}^*_k$ is the limit of the nested subspace algorithm:

$$\begin{aligned} &\mathcal{W}^*_{k,0} = \mathbb{R}^n \\ &\vdots \\ &\mathcal{W}^*_{k,(j+1)} = A_0^{-1} \left(E \mathcal{W}^*_{k,(j)} \right)_{A_1 = 0}, \quad j \ge 0. \end{aligned} \tag{7}$$

Moreover, if $\phi(t) \in \mathcal{W}^*_{cont}\ \forall t \in [-\tau, 0]$ then $\mathbf{x}(t) \in \mathcal{W}^*_k\ \forall t > \tau$ and $(\lambda E - A_0)$ is invertible for some $\lambda \in \mathbb{C}$ (condition for uniqueness), then $\mathcal{W}^*_k \cap \aleph(E) = \{0\}$.

2. Necessary Definitions and Lemmas

Definition 1. [26] *The matrix pair* $(E,\, A_0)$ *is said to be regular if* $\det(sE - A_0)$ *is not identically zero.*

Definition 2. [26] *The matrix pair* $(E,\, A_0)$ *is said to be impulse free if* $\deg(\det(sE - A_0)) = \operatorname{rank} E$.

Definition 3. *The linear continuous singular time delay system* (5) *is said to be regular and impulse free, if the matrix pair* $(E,\, A_0)$ *is regular and impulse free.*

The linear continuous singular time delay system (5) may have an impulsive solution, however, the regularity and the absence of impulses of the matrix pair $(E,\ A_0)$ ensure the existence and uniqueness of an impulse free solution to the system under consideration, which is defined in the following *Lemma.*

Lemma 1. [26] *Suppose that the matrix pair* $(E,\, A_0)$ *is regular and impulse free and unique on* $[0,\, \infty[$.

In the next section, we use stability definitions to derive the main result of this study.

3. Stability Definitions and Main Results

Stability plays a central role in the theory of systems and control engineering. There are different kinds of stability problems that arise in the study of dynamic systems, such as *Lyapunov* stability, finite time stability, practical stability, technical stability and BIBO stability. The first part of this section deals with the asymptotic stability of the equilibrium points of *linear continuous singular systems.* Since we consider linear systems, the latter one is equivalent to the study of the stability of systems.

The *Lyapunov* direct method (LDM) is widely covered in a number of very well-known references. Here we present some different and interesting approaches to this problem, mostly based on the contributions of the authors of this paper.

Definition 4. *The linear continuous singular time delay system* (5) *is said to be stable, if for any* $\varepsilon > 0$ *there exists a scalar* $\delta(\varepsilon) > 0$*, such that for any compatible initial conditions* $\phi(t)$*, satisfying* $\sup\limits_{-\tau \le t \le 0} \|\phi(t)\| \le \delta(\varepsilon)$*, solution* $\mathbf{x}(t)$ *to the system under consideration fulfills* $\|\mathbf{x}(t)\| \le \varepsilon \,\forall t \ge 0$*. Moreover if* $\lim\limits_{t\to\infty} \|\mathbf{x}(t)\| \to 0$*, system* (5) *is said to be asymptotically stable* [26].

Due to the system structure and complicated solution, the regularity of the systems is the condition to make the solution to singular control systems exist and be unique. Moreover, if the consistent initial conditions are applied, then the closed form of solutions can be established.

The new result presented here is based mostly on the result given in [24] and [35]. For some other important details, used in paper, [31–35].

Lemma 2. *Let there be given control system* (5)*. Scalar aggregate function* $V(\mathbf{y}(t))$ *is defined as:*

$$V(\mathbf{y}(t)) = V(\mathbf{y}^*(t)) = \mathbf{y}^\top(t)\, P\, \mathbf{y}(t) = \mathbf{y}^{*T}(t)\, E^\top P\, E\, \mathbf{y}^*(t)\,, \tag{8}$$

where vector $y(t)$ *has the form:*

$$\begin{gathered}\mathbf{y}(t)=E\,\mathbf{y}^*(t),\\ \mathbf{y}^*(t)=\mathbf{x}(t)+\int_0^\tau H(\theta)\,\mathbf{x}(t-\theta)\,d\theta,\\ \mathbf{y}(t)=E\,\mathbf{y}^*(t)=E\mathbf{x}(t)+E\int_0^\tau H(\theta)\,\mathbf{x}(t-\theta)\,d\theta=\\ =E\left(\mathbf{x}(t)+\int_0^\tau H(\theta)\,\mathbf{x}(t-\theta)\,d\theta\right),\end{gathered}\tag{9}$$

where

1) $H(\theta)$ *is a continuous and differentiable* $(n\times n)$*-matrix over time interval* $[0,\tau]$ *satisfying following differential matrix equation*

$$E\,\dot{H}(\vartheta)=(A_0+EH(0))\,H(\vartheta)\,,\quad \vartheta\in[0,\ \tau]\tag{10}$$

with initial condition

$$E\,H(\tau)=A_1;\tag{11}$$

2) P *is a symmetric positive definite matrix, eg.* $P=P^\top>0$.
Then, the Euler derivatives of $V(\mathbf{y}^*(t))$ *along the trajectories of* (5) *are given as:*

$$\dot{V}(\mathbf{y}^*(t))=\mathbf{y}^{*\top}(t)\,\Pi\,\mathbf{y}^*(t)\,,\tag{12}$$

where

$$\Pi=E^\top P\,(A_0+PEH(0))+(A_0+PEH(0))^\top\,PE.\tag{13}$$

Proof. Based on the definition of $V(\mathbf{y}(t))$, it follows that its derivative with respect to t has the form:

$$\begin{gathered}\dot{V}(\mathbf{y}(t))=\dot{\mathbf{y}}^\top(t)\,P\mathbf{y}(t)+\mathbf{y}^\top(t)\,P\dot{\mathbf{y}}(t)=\\ =\left(E\dot{\mathbf{x}}(t)+\frac{d}{dt}E\int_0^\tau H(\theta)\,\mathbf{x}(t-\theta)\,d\theta\right)^\top P\left(E\mathbf{x}(t)+E\int_0^\tau H(\eta)\,\mathbf{x}(t-\eta)\,d\eta\right)+\\ +\left(E\mathbf{x}(t)+E\int_0^\tau H(\theta)\,\mathbf{x}(t-\theta)\,d\theta\right)^\top P\left(E\dot{\mathbf{x}}(t)+\frac{d}{dt}E\int_0^\tau H(\eta)\,\mathbf{x}(t-\eta)\,d\eta\right)=\\ =\left(E\dot{\mathbf{x}}+E\left(\int_0^\tau \dot{H}(\theta)\,\mathbf{x}(t-\theta)\,d\theta-H(\tau)\,\mathbf{x}(t-\tau)+H(0)\,\mathbf{x}(t)\right)\right)^\top\times\\ \times P\left(E\mathbf{x}(t)+E\int_0^\tau H(\eta)\,\mathbf{x}(t-\eta)\,d\eta\right)+\left(\mathbf{x}^\top(t)\,E^\top+\left(\int_0^\tau \mathbf{x}^\top(t-\theta)\,H^\top(\theta)\,d\theta\right)E^\top\right)\times\\ \times P\left(E\dot{\mathbf{x}}(t)+E\left(\int_0^\tau \dot{H}(\eta)\,\mathbf{x}(t-\eta)\,d\eta-H(\tau)\,\mathbf{x}(t-\tau)+H(0)\,\mathbf{x}(t)\right)\right),\end{gathered}\tag{14}$$

whence it immediately follows that

$$\dot V(\mathbf{y}(t)) = \left(\dot{\mathbf{x}}^\top(t)E^\top + \left(\int_0^\tau \mathbf{x}^\top(t-\theta)\dot H^\top(\theta)\,d\theta\right)E^\top - \mathbf{x}^\top(t-\tau)H^\top(\tau)E^\top + \right.$$
$$\left. + \mathbf{x}^\top(t)H^\top(0)E^\top\right)\times P\left(E\mathbf{x}(t) + E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)\,d\eta\right) +$$
$$+\left(\mathbf{x}^\top(t)E^\top + \left(\int_0^\tau \mathbf{x}^\top(t-\theta)H^\top(\theta)\,d\theta\right)E^\top\right)\times \tag{15}$$
$$\times P\left(E\dot{\mathbf{x}}(t) + E\left(\int_0^\tau \dot H(\eta)\mathbf{x}(t-\eta)\,d\eta - H(\tau)\mathbf{x}(t-\tau) + H(0)\mathbf{x}(t)\right)\right).$$

Since

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \\ \dot{\mathbf{x}}^\top(t)E^\top &= \mathbf{x}^\top(t)A_0^\top + \mathbf{x}^\top(t-\tau)A_1^\top, \\ E\,\dot H(\vartheta) &= (A_0 + EH(0))H(\vartheta), \quad \vartheta\in[0,\ \tau], \\ \left(E\,\dot H(\vartheta)\right)^\top &= \dot H^\top E^\top(\vartheta) = H^\top(\vartheta)(A_0 + EH(0))^\top, \\ E\,H(\tau) &= A_1,\ H^\top(\tau)E^\top = A_1^\top, \end{aligned} \tag{16}$$

we have:

$$\dot V(\mathbf{y}(t)) = \left(\dot{\mathbf{x}}^\top(t)E^\top + \left(\int_0^\tau \mathbf{x}^\top(t-\theta)\dot H^\top(\theta)\,d\theta\right)E^\top - \mathbf{x}^\top(t-\tau)H^\top(\tau)E^\top + \right.$$
$$\left. + \mathbf{x}^\top(t)H^\top(0)E^\top\right)\times P\left(E\mathbf{x}(t) + E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)\,d\eta\right) +$$
$$+\left(\mathbf{x}^\top(t)E^\top + \left(\int_0^\tau \mathbf{x}^\top(t-\theta)H^\top(\theta)\,d\theta\right)E^\top\right)\times$$
$$\times P\left(E\dot{\mathbf{x}}(t) + E\left(\int_0^\tau \dot H(\eta)\mathbf{x}(t-\eta)\,d\eta - H(\tau)\mathbf{x}(t-\tau) + H(0)\mathbf{x}(t)\right)\right)$$

or:

$$\dot V(\mathbf{y}(t)) =$$
$$= \left(\left(\mathbf{x}^\top(v)A_0^\top + \mathbf{x}^\top(t-\tau)A_1^\top\right) + \left(\int_0^\tau \mathbf{x}^\top(t-\theta)H^\top(\vartheta)(A_0 + EH(0))^\top\,d\theta\right) - \right.$$
$$\left. - \mathbf{x}^\top(t-\tau)H^\top(\tau)E^\top + \mathbf{x}^\top(t)H^\top(0)E^\top\right)\times P\left(E\mathbf{x}(t) + E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)\,d\eta\right) +$$
$$+\left(\mathbf{x}^\top(t)E^\top + \left(\int_0^\tau \mathbf{x}^\top(t-\theta)H^\top(\theta)\,d\theta\right)E^\top\right)\times P\left((A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)) + \right.$$
$$\left. + \left(\int_0^\tau (A_0 + EH(0))H(\eta)\mathbf{x}(t-\eta)\,d\eta - A_1\mathbf{x}(t-\tau) + EH(0)\mathbf{x}(t)\right)\right).$$

If we introduce:

$$\dot V(\mathbf{y}(t)) = \dot V_1(\mathbf{y}(t)) + \dot V_2(\mathbf{y}(t)), \tag{17}$$

then, to make it easier to calculate, one can get

$$\dot{V}_2(\mathbf{y}(t)) = \left(\mathbf{x}^\top(t)E^\top P + \left(\int_0^\tau \mathbf{x}^\top(t-\theta)Q^\top(\theta)d\theta\right)E^\top P\right) \times \left(A_0\mathbf{x}(t)+\right.$$
$$\left. +A_1\mathbf{x}(t-\tau) + \left(\int_0^\tau (A_0 + EH(0))H(\eta)\mathbf{x}(t-\eta)d\eta - A_1\mathbf{x}(t-\tau) + EH(0)\mathbf{x}(t)\right)\right), \tag{18}$$

$$\dot{V}_1(\mathbf{y}(t)) = \begin{pmatrix} \mathbf{x}^\top(t)A_0^\top PE\mathbf{x}(t) + \mathbf{x}^\top(t-\tau)A_1^\top PE\mathbf{x}(t) + \\ +\int_0^\tau \mathbf{x}^\top(t-\theta)H^\top(\vartheta)(A_0 + EH(0))^\top PE\mathbf{x}(t)d\theta - \\ -\mathbf{x}^\top(t-\tau)H^\top(\tau)E^\top PE\mathbf{x}(t) + \mathbf{x}^\top(t)H^\top(0)E^\top PE\mathbf{x}(t) \end{pmatrix} \times$$
$$\times \begin{pmatrix} \mathbf{x}^\top(t)A_0^\top P \times E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)d\eta + \\ +\mathbf{x}^\top(t-\tau)A_1^\top P \times E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)d\eta \times E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)d\eta + \\ +\int_0^\tau \mathbf{x}^\top(t-\theta)H^\top(\theta)(A_0 + EH(0))^\top Pd\theta \times E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)d\eta - \\ -\mathbf{x}^\top(t-\tau)H^\top(\tau)E^\top P \times E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)d\eta + \\ +\mathbf{x}^\top(t)H^\top(0)E^\top P \times E\int_0^\tau H(\eta)\mathbf{x}(t-\eta)d\eta \end{pmatrix}. \tag{19}$$

After some rearrangement it leads to:

$$\begin{aligned} &\dot{V}(\mathbf{y}^*(t)) = \mathbf{y}^{*T}(t)\,\Pi\,\mathbf{y}^*(t) = \\ &= \mathbf{y}^{*T}(t)E^\top P\left((A_0 + PEH(0))\right) + \left((A_0 + PEH(0))^\top\right) PE\,\mathbf{y}^*(t). \end{aligned} \tag{20}$$

□

Now, we are in the position to present our main result. The theorem below presents the stability result for the singular time delayed system, described by (1), and is the result of application of the second Lyapunov method.

Theorem 1. *Let the regular singular time delay system be described as in* (1)*:*

$$\begin{array}{ll} (a) & E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \\ (b) & \mathbf{x}(t) = \phi(t), \quad -\tau \le t \le 0, \\ (c) & \mathbf{x}_0 \in \mathcal{W}_k \backslash \{0\}. \end{array} \tag{21}$$

Under the assumption that algebraic system of equations in (21a) is purely algebraic, e.g. without delays, the system, given by (21.a) and accompanied by (21b, 21c), is asymptotically stable, if and only if:

1) A_0 is an invertible matrix;

2) there exists a symmetric positive definite matrix P, such that

$$E^\top P\left((A_0 + PEH(0))\right) + \left((A_0 + PEH(0))^\top\right) PE = -Q, \tag{22}$$

where Q is symmetric, positive definite in the sense that

$$\mathbf{y}^{*T}(t)Q\,\mathbf{y}^*(t) > 0 \text{ for all } \mathbf{x}(t) \in \mathcal{W}_k \backslash \{0\}, \tag{23}$$

and for $t \in [0\ \ \tau]$ the matrix $H(\tau)$ satisfies

$$E\,\dot{H}(\vartheta) = (A_0 + EH(0))\,H(\vartheta)\,, \quad \vartheta \in [0,\ \tau] \tag{24}$$

with initial condition:

$$E\,H(\tau) = A_1 \tag{25}$$

and

$$H(\tau) = 0 \tag{26}$$

elsewhere. $H(0)$ is the solution to the following equation:

$$\left(\hat{E}^D\hat{E} - I\right)H(0) = 0, \quad \hat{E} = (\lambda E - A_0)^{-1}\,,\ for\ some\ \lambda \in \mathbb{R}. \tag{27}$$

Proof. To prove sufficiency, note that [24]

$$\mathcal{W}_k \cap \mathbb{N}(E) = \{0\}\,, \tag{28}$$

so

$$V(\mathbf{y}^*(t)) = \mathbf{y}^{*T}(t)\,E^\top P\,E\,\mathbf{y}^*(t) \tag{29}$$

is a positive definite quadratic form on $\mathcal{W}_k$.

Under the condition given by (28), $\mathbf{x}_0$ is a consistent initial condition, iff $\mathbf{x}_0 \in \mathcal{W}_k$. Moreover, $\mathbf{x}_0$ generates a unique solution $\mathbf{x}(t) \in \mathcal{W}_{k^*}\,(t \geq 0)$, which is real analytical on $(t,\ t \geq 0)$, [24]. Therefore, all smooth solutions $\mathbf{x}(t)$ belong to $\mathcal{W}_{k^*}$, and $V(\mathbf{y}^*(t))$ can be used as the Lypunov function.

To prove necessity, note that

$$\mathbf{x}(t) \in \mathcal{W}_{k^*}\,(t \geq 0)\,, \tag{30}$$

if

$$\left(E_0^D E_0 - I\right)\mathbf{x}_0 = \mathbf{0}, \tag{31}$$

where the superscript "D" denotes the Drazin inverse and by applying the results of [1], we get

$$\mathbf{y}^{*T}(t)\,\tilde{Q}\ \ \mathbf{y}^*(t) > 0$$

for all $\mathbf{x}(t) \in \mathcal{W}_k \backslash \{0\}$, because $\tilde{Q}$ is positive definite and satisfies

$$\tilde{E}^\top\tilde{P} + \tilde{P}\tilde{E} = \tilde{Q}, \tag{32}$$

where

$$\begin{gathered} \tilde{E} = E_0 + \varepsilon\left(E_0^D E_0 - I\right), \\ \varepsilon > 0, \\ E_0 = A_1^{-1}E, \end{gathered} \tag{33}$$

and $\tilde{P}$ is a symmetric positive definite solution to the above equation.

□

Remark 2. If we have a time delay system only (cf. [25]), we have to solve a very complicated nonlinear transcendental matrix equation:

$$e^{(A_0 + H(0))\tau}H(0) = A_1 \tag{34}$$

with respect to $H(0)$ in order to check asymptotic stability of system under consideration. In original paper [25], a crucial mistake was made when solving (3) and using no representative matrix for system stability testing in the corresponding Lyapunov matrix equation. The paper [20] gave an adequate counterexample the basic theorem and it was demonstrated that their statement is incorrect. To eliminate this error, we was proposed a new theorem formulation and showed how to correct the example given there.

Remark 3. The equation can be addressed using the fact that $H(0)$ in (10) must satisfy demand for consistent initial conditions to avoid impulsive solutions:

$$\left(\hat{E}^D\hat{E} - I\right) H(0) = 0, \tag{35}$$

where

$$\hat{E} = (\lambda E - A_0)^{-1}, \text{ for some } \lambda \in \mathbb{R}. \tag{36}$$

Therefore, there is no need to solve a nonlinear transcendental matrix algebraic equation.

Conclusion

Generally, this paper extends some of the basic results in the area of the Lyapunov stability to linear continuous singular time delay systems. A part of these results is hence a geometric counterpart of algebraic theory of [1] enhanced with the appropriate criteria to cover the need for asymptotic system stability under the presence of the actual time delay term. We consider the geometric description of consistent initial conditions that generate tractable and smooth solutions to such problems and analyze the construction of Lyapunov stability for this class of systems. Testing the definiteness of the particular quadratic form on subspace of consistent initial conditions can be a very complicated numerical task, but it can be addressed by the approach based on the controllability and observability test of systems matrices, first time presented in [2]. Some other aspects of singular time delay systems, including different stability concepts, can be found in [27–31].

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СОВМЕСТНОСТЬ И УСТОЙЧИВОСТЬ ПО ЛЯПУНОВУ ЛИНЕЙНЫХ ВЫРОЖДЕННЫХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ: ГЕОМЕТРИЧЕСКИЙ ПОДХОД

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На практике при рассмотрении различных систем с управлением (химические инженерные системы, линии передачи без потерь, крупномасштабное управление электрической сетью, управление ориентацией самолета, гибкое управление руками роботов и т.д.) часто во многих ситуациях мы можем наблюдать наличие временного запаздывания. Вырожденные системы с запаздыванием являются динамическими системами, описываемыми взаимосвязанными алгебраическими и дифференциальными уравнениями. В данной работе исследуются геометрические представления начальных данных, которые обеспечивают гладкость решений рассматриваемых задач. Также изучается построение теории устойчивости Ляпунова для ограничения скорости убывания решений. Для одного класса изучаемых систем получены новые условия асимптотической устойчивости, зависящие от запаздывания. Более того, результат выражается в терминах матриц, которые задают систему и естественным образом возникают при моделировании, однако при этом отсутствует необходимость введения алгебраических преобразований в утверждение основной теоремы.

Ключевые слова: вырожденные системы с запаздыванием; устойчивость по Ляпунову; совместные начальные условия.

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