

## ON A NONLINEAR PROBLEM OF THE BREAKING WATER WAVES

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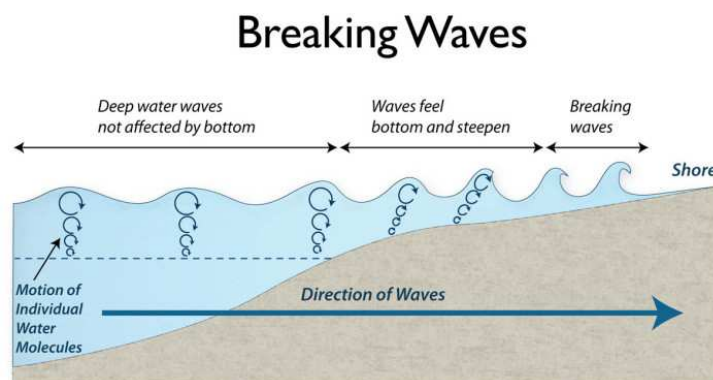
The paper is devoted to the initial boundary value problem for the Korteweg-de Vries–Benjamin–Bona–Mahony equation in a finite domain. This particular problem arises from the phenomenon of long wave with small amplitude in fluid. For certain initial-boundary problems for the Korteweg-de Vries–Benjamin–Bona–Mahony equation, we obtain the conditions of blowing-up of global and travelling wave solutions in finite time. The proof of the results is based on the nonlinear capacity method. In closing, we provide the exact and numerical examples.

*Keywords:* breaking waves; Korteweg-de Vries–Benjamin–Bona–Mahony equation; blow-up of solution; initial-boundary problems.

## Introduction

### Breaking Waves

In fluid dynamics, a breaking wave (see Fig. 1) is a wave whose amplitude reaches a critical level at which some process can suddenly start to occur that causes large amounts of wave energy to be transformed into turbulent kinetic energy. At this point, simple physical models that describe wave dynamics often become invalid, particularly those that assume linear behaviour.



**Fig. 1.** Breaking waves

Breaking of water surface waves may occur anywhere that the amplitude is sufficient, including in mid-ocean. However, it is particularly common on beaches because wave heights are amplified in the region of shallower water (because the group velocity is lower there). There are four basic types of breaking water waves [1]. They are spilling, plunging, collapsing, and surging (see Fig. 2).

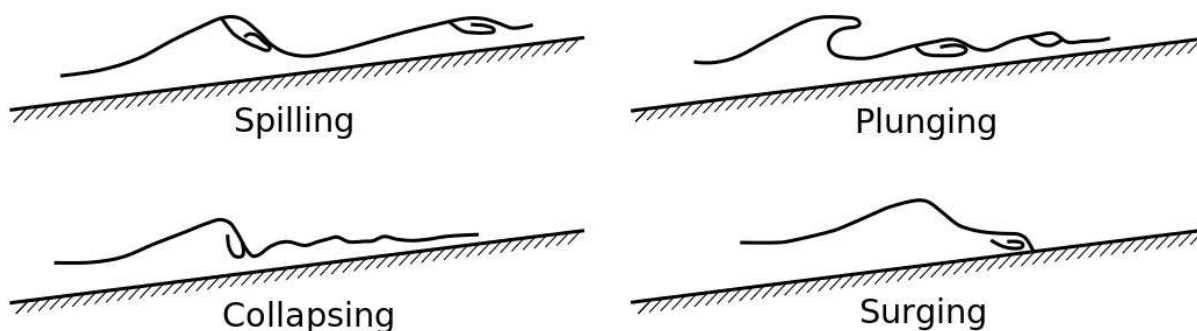


Fig. 2. Types of breaking water waves

### History of Problem

The Korteweg-de Vries equation and the Benjamin–Bona–Mahony equation are two typical examples associated with the effects of dissipation, dispersion, nonlinearity and also provide a description of the propagation of waves with small amplitude in water or solution in other liquid medium. The Korteweg-de Vries equation is described as follows:

$$u_t + u_{xxx} + u_x + uu_x = 0.$$

The Benjamin–Bona–Mahony equation is an alternative to the Korteweg-de Vries equation [2] which is described as follows:

$$u_t - u_{txx} + u_x + uu_x = 0.$$

Francius, Pelinovsky and Slunyaev introduced the wave dynamics of the following equation [3]:

$$u_t - u_{txx} + \alpha u_{xxx} + \beta u_x + uu_x = 0, \quad (1)$$

where  $\alpha, \beta \in \mathbb{R}$ . The equation (1) is so-called Korteweg-de Vries–Benjamin–Bona–Mahony equation.

### Statement of the Problem

In this paper, we consider one of the mathematical problem of the breaking water waves, the Korteweg-de Vries–Benjamin–Bona–Mahony equation with initial conditions described as follows:

$$u_t - u_{txx} + \alpha u_{xxx} + \beta u_x + uu_x = 0, \quad t > 0, \quad x \in (0, L), \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in [0, L], \quad (3)$$

where  $\alpha, \beta \in \mathbb{R}$  and  $u_0$  is given function.

The Korteweg-de Vries–Benjamin–Bona–Mahony equation has important application in different physical situations such as waves on shallow water, and processes in semiconductors with differential conductivity. In [5] Korpusov and Yushkov obtained sufficient conditions for the finite time blow-up of solutions of time and space initial problems for Korteweg-de Vries–Benjamin–Bona–Mahony type equation in the form

$$u_t - u_{txx} - u_{xxx} + uu_x = 0.$$

They used the powerful method of nonlinear capacity, developed by S.I. Pohozaev [10].

We also note that the blowing-up of solutions of the initial problems for the Korteweg-de Vries and critical Korteweg-de Vries equations are investigated in [6–9, 13–15]. Recently Li and Liu [4] studied the global and local (with respect to  $t > 0$ ) solvability of the problem (2), (3) with boundary conditions

$$\begin{aligned} u(0, t) &= h_1(t), \\ u(L, t) &= h_2(t), \\ u_x(L, t) &= h_3(t), \quad t \geq 0. \end{aligned}$$

This paper is devoted to singular solutions of the problem (2), (3), more precisely, to solutions that blow up in a finite time. The approach to the problem is based on the method of nonlinear capacity [10, 11], more precisely, on the choice of test functions corresponding to initial and boundary conditions under consideration.

Here, we give a simplest case of the analysis of a “rough” blow-up, i.e., the case where the solution tends to infinity as  $t > T$  on  $[0, L]$  of values  $x$ , more exactly, when the integral

$$\int_0^L u(t, x)\varphi(x)dx$$

tends to infinity as  $t > T$  for the given test function  $\varphi$ .

## 1. Main Result

We consider a test function  $\varphi \in C^3([0, L])$  and monotonically nondecreasing:

$$\varphi'(x) \geq 0 \quad \text{for } x \in [0, L], \tag{4}$$

and let the function  $\varphi$  satisfy the following properties:

$$\begin{cases} \theta_1 := \int_0^L \frac{(\alpha\varphi''' + \beta\varphi')^2}{\varphi'} dx < \infty; \\ \theta_2 := \int_0^L \frac{(\varphi - \varphi'')^2}{\varphi'} dx < \infty. \end{cases} \tag{5}$$

Suppose the classical solution  $u(x, t) \in C_{t,x}^{1,3}((0, L) \times (0, T))$ .

Multiply the Korteweg-de Vries–Benjamin–Bona–Mahony equation (2) by a test function  $\varphi$  we have

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^L (u(x, t) - u_{xx}(x, t))\varphi(x)dx &= -\alpha \int_0^L u_{xxx}(x, t)\varphi(x)dx - \\ &\quad - \beta \int_0^L u_x(x, t)\varphi(x)dx - \int_0^L u_x(x, t)u(x, t)\varphi(x)dx. \end{aligned}$$

Integrating by parts the last equation we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^L u(x, t)(\varphi(x) - \varphi''(x))dx &= \alpha \int_0^L u(x, t)\varphi'''(x)dx + \beta \int_0^L u(x, t)\varphi'(x)dx + \\ &+ \frac{1}{2} \int_0^L u^2(x, t)\varphi'(x)dx + \mathcal{B}(u(x, t), \varphi(x)) \Big|_{x=0}^{x=L}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \mathcal{B}(u(x, t), \varphi(x)) &= u_{tx}(x, t)\varphi(x) - u_t(x, t)\varphi'(x) - \alpha u_{xx}(x, t)\varphi(x) + \alpha u_x(x, t)\varphi'(x) - \\ &- \alpha u(x, t)\varphi''(x) - \beta u(x, t)\varphi(x) - \frac{1}{2}u^2(x, t)\varphi(x). \end{aligned}$$

Then, using properties (4), we find

$$\begin{aligned} \int_0^L (2u(x, t) (\alpha\varphi'''(x) + \beta\varphi'(x)) + u^2(x, t)\varphi'(x)) dx &= \\ = \int_0^L \left( u(x, t) + \frac{\alpha\varphi'''(x) + \beta\varphi'(x)}{\varphi'(x)} \right)^2 \varphi'(x) dx - \int_0^L \frac{(\alpha\varphi'''(x) + \beta\varphi'(x))^2}{\varphi'(x)} dx. \end{aligned}$$

We denote by  $w(x, t)$  the following function

$$w(x, t) = u(x, t) + \frac{\alpha\varphi'''(x) + \beta\varphi'(x)}{\varphi'(x)}.$$

Let

$$F(t) = \int_0^L w(x, t)(\varphi(x) - \varphi''(x))dx.$$

By using the Hölder inequality for  $F(t)$ , we obtain the following estimate

$$\left( \int_0^L w(x, t)(\varphi(x) - \varphi''(x))dx \right)^2 \leq \int_0^L w^2(x, t)\varphi'(x)dx \int_0^L \frac{(\varphi(x) - \varphi''(x))^2}{\varphi'(x)} dx.$$

Therefore, using the properties of the test function (5) for the expression (6), we obtain the following first order differential inequality

$$F'(t) \geq \frac{\theta_2^{-1}}{2} F^2(t) + \Phi(t) - \frac{\theta_1}{2} \tag{7}$$

with initial condition

$$F(0) = \int_0^L \left( u_0(x) + \frac{\alpha\varphi'''(x) + \beta\varphi'(x)}{\varphi'(x)} \right) \varphi(x) dx,$$

where  $\Phi(t) = \mathcal{B}(u(L, t), \varphi(L)) - \mathcal{B}(u(0, t), \varphi(0))$ . Then the following results are true

**Theorem 1.** *Let  $u_0(x) \in L^1([0, L])$  and the solution  $u \in C_{t,x}^{1,3}((0, L) \times (0, T))$  of the equation (2) be such that there exists a function  $\varphi$  satisfying conditions (4), (5) such that*

$$\Phi(t) \geq \sigma, \text{ for all } t > 0,$$

where  $\sigma$  is a some constant. Then

(A) if  $\sigma > \theta_2$ , then  $F(t) \rightarrow +\infty$  for  $t \rightarrow T_1^*$ , where

$$T_1^* = \frac{2\sqrt{\alpha}}{\sqrt{\sigma - \theta_2}} \left( \frac{\pi}{2} - \arctan \frac{F(0)}{2\sqrt{\theta_1(\sigma - \theta_2)}} \right);$$

(B) if  $\sigma = \theta_2$  and  $F(0) > 0$ , then  $F(t) \rightarrow +\infty$  for  $t \rightarrow T_2^*$ , where  $T_2^* = \frac{4\theta_1}{F(0)}$ ;

(C) if  $\sigma < \theta_2$  and  $F(0) > 2\sqrt{\theta_1(\theta_2 - \sigma)}$ , then  $F(t) \rightarrow +\infty$  for  $t \rightarrow T_3^*$ , where

$$T_3^* = \frac{\sqrt{\theta_1}}{\sqrt{\theta_2 - \sigma}} \ln \frac{F(0) + 2\sqrt{\theta_1(\theta_2 - \sigma)}}{F(0) - 2\sqrt{\theta_1(\theta_2 - \sigma)}}.$$

Applying the theory of ordinary differential inequalities, Theorem 1 can be proved.

### 1.1. Examples

**Example 1.** Note that the trial function method has great practical convenience. For example, if in problem (2), (3) on the interval  $[0, 1]$  are given boundary conditions

$$\begin{aligned} u(0, t) = 0, \quad u(1, t) = 0, \\ u_{xx}(1, t) - u_{tx}(1, t) - 5u_x(1, t) = 0, \quad t \geq 0. \end{aligned}$$

Then, if  $\varphi(x) = x^5$  we obtain  $\theta_1 := 741$ ,  $\theta_2 := \frac{359}{5}$  and  $\Phi(t) = 0$ , for all  $t > 0$ . Hence it follows from Theorem 1 that, under condition

$$\int_0^1 u_0(x)x^5 dx > \frac{2\sqrt{1330095}}{5}$$

the solution of problem (2), (3) blows up in finite time.

**Example 2.** Let  $\alpha = 0$  in problem (2), (3) on the interval  $[0, 1]$  and let the solution of problem (2), (3) satisfy the boundary conditions

$$\begin{aligned} u_x(0, t) = 0, \quad u_x(1, t) = 0, \\ u^2(1, t) + 2\beta u(1, t) = 0, \quad t \geq 0. \end{aligned}$$

Then, if  $\varphi(x) = x$  we obtain  $\theta_1 := \beta^2$ ,  $\theta_2 := \frac{1}{3}$  and  $\Phi(t) = 0$ , for all  $t > 0$ . Hence it follows from Theorem 1 that, under condition

$$\int_0^1 u_0(x)x dx > \frac{2\sqrt{3}\beta}{3}$$

the solution of problem (2), (3) blows up in finite time.

**Example 3.** Let in problem (2), (3) on the interval  $[0, L]$  are given boundary conditions

$$\begin{aligned} u(0, t) = 0, \quad u_x(0, t) = 0, \\ u_{xx}(0, t) + u_x(0, t) + u_t(L, t) = 0, \quad t \geq 0. \end{aligned}$$

Then, taking a function  $\varphi(x) = (x - L)$  we obtain  $\theta_1 := L$ ,  $\theta_2 := \frac{L^3}{3}$  and  $\Phi(t) = 0$ , for all  $t > 0$ . Hence it follows from Theorem 1 that, under condition

$$\int_0^L u_0(x)(x - L)dx > \frac{2\sqrt{3}L^2}{3}$$

the solution of problem (2), (3) blows up in finite time.

**Example 4.** Let  $\beta = 0$  in the problem (2), (3) with

$$u_0(x) = \mathcal{A}x, \mathcal{A} \neq 0$$

on the interval  $[0, 1]$ . The solution of this problem is the function

$$u(x, t) = \frac{x}{T^* - t},$$

where  $T^* = 1/\mathcal{A}$ .

Thus, for  $\mathcal{A} > 0$  the solution of this problem becomes  $+\infty$  as  $t \rightarrow T^*$ . On the other hand, for  $\mathcal{A} < 0$  there exists a global (for all  $t > 0$ ) smooth solution of problem (2), (3).

We built the graph of the solution for  $\mathcal{A} = 1/66$ . As can be seen from Fig. 3, the solution of the problem (2), (3) with  $\mathcal{A} = 1/66$  has a discontinuity at  $t = 66$ .

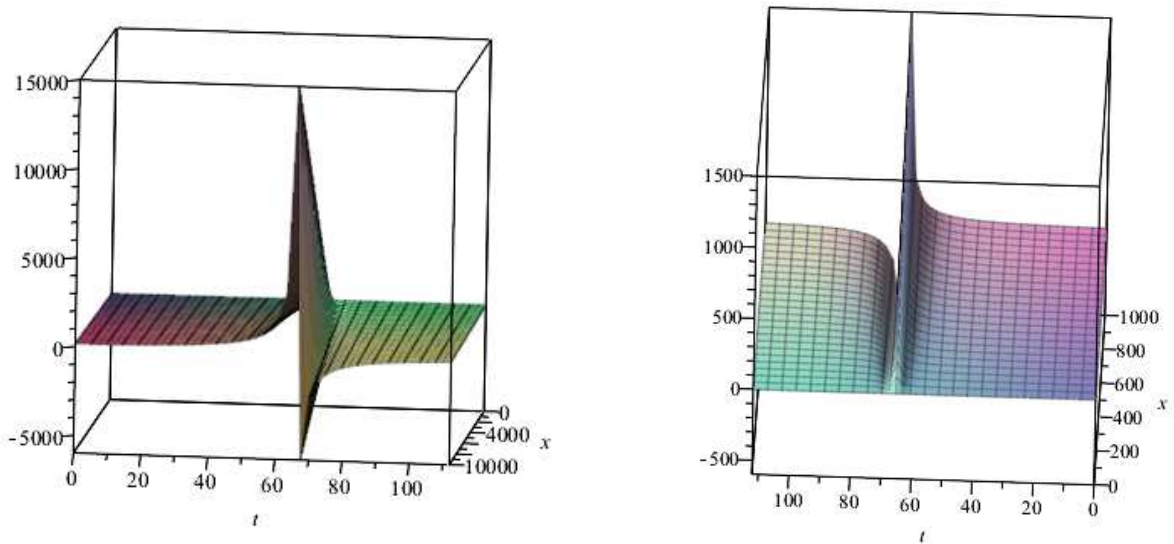


Fig. 3. Graphs of  $u(x, t)$  for  $\mathcal{A} = 1/66$

## 2. Singular Travelling Wave Solutions

We consider the traveling wave type solutions of the Korteweg-de Vries–Benjamin–Bona–Mahony equation (1) with  $\alpha = -1$ ,  $\beta = 1$ :

$$u(x, t) = u(\xi),$$

where  $\xi = x - ct$  and  $c$  is the wave velocity. Then  $u(\xi)$  satisfies

$$(c - 1)u''' - (c - 1)u' - uu' = 0. \tag{8}$$

Equation (8) admits the following integrals:

$$(c - 1)u'' - (c - 1)u - \frac{u^2}{2} - C_0 = 0. \tag{9}$$

Here  $C_0$  is an arbitrary constant. A solution of (9) is a function  $u \in L^2_{loc}(I)$ ,  $I \subset \mathbb{R}$  that satisfies the integral identity

$$\int_I u^2 \phi d\xi = 2(c - 1) \int_I u(\phi'' - \phi) d\xi - 2C_0 \int_I \phi d\xi, \quad \phi \in C^2_0(I). \tag{10}$$

### 2.1. Nonexistence of Travelling Wave Solutions

The following results are true

**Theorem 2.** *The equation (8) with support  $L \geq 2$ , satisfying the inequality*

$$C_0 > \frac{(c - 1)^2 b}{2dL^2}, \tag{11}$$

*does not admit a solution.*

*Proof.* We multiply equation (9) by a nonnegative test function  $\phi \in C^2_0(\mathbb{R})$  with compact support. Then after integration we obtain (10). Hence, by the Young inequality with parameter  $a > 0$ , we find that

$$\int_I u^2 \phi d\xi \leq \frac{(c - 1)}{a} \int_I u^2 \phi d\xi + a(c - 1) \int_I \frac{(\phi'' - \phi)^2}{\phi} d\xi - 2C_0 \int_I \phi d\xi. \tag{12}$$

We now take the test function:

$$\phi(\xi) = \phi_0(\eta), \quad \eta = \frac{\xi}{L},$$

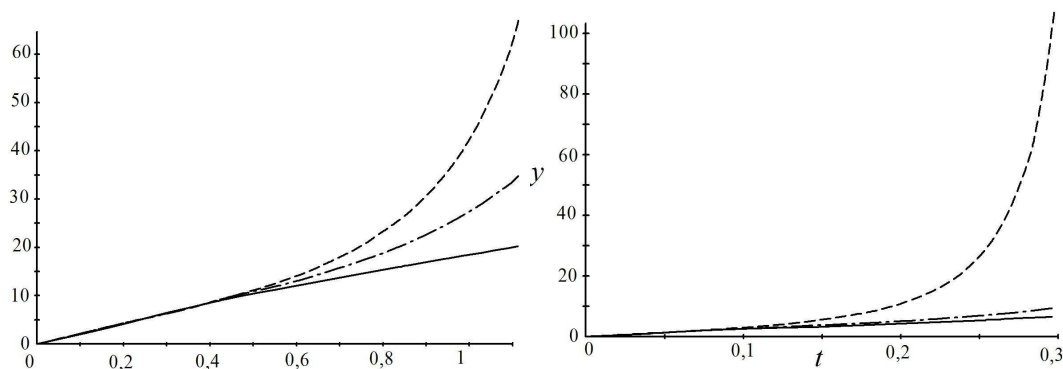
where  $L \geq 2$  is a free parameter and the function  $0 \leq \phi \in C^2(I)$  such that

$$\begin{aligned} \phi_0(\eta) &= \begin{cases} 1, & \text{if } |\eta| \leq 1, \\ 0, & \text{if } |\eta| \geq 2, \end{cases} \\ b &= \int_{-2}^2 \frac{|\phi''_0(\eta) - L^2 \phi_0(\eta)|^2}{\phi_0(\eta)} d\eta < \infty. \end{aligned} \tag{13}$$

Then, if  $a = c - 1$  the inequality (12) implies

$$(c - 1)^2 \frac{b}{L^2} \geq 2C_0 d, \quad d = \int_{-2}^2 \phi_0(\eta) d\eta. \tag{14}$$

From this it directly follows that if there exist  $C_0$  such that the inequality (12) holds, then there is no such bounded traveling wave solution of equation (10). □

Fig. 4. Graphs of  $u(\xi)$ 

## 2.2. Numerical Examples

In this subsection we consider some numerical examples for equation (9) with different viscosities. Suppose that we know some initial data for a traveling wave. Then it can be seen from Fig. 4, that the time for the blows-up of traveling waves increases if the wave velocity is decreasing.

## Conclusion

The present paper is devoted to the initial problem for the Korteweg-de Vries–Benjamin–Bona–Mahony equation in a finite interval. This particular problem arises from the phenomenon of long breaking waves with small amplitude in fluid. For certain initial-boundary problems for the Korteweg-de Vries–Benjamin–Bona–Mahony equation, we obtain conditions of blowing-up of local solutions in finite time. Proofs of the results are based on the nonlinear capacity method. We also proved the existence of the singular travelling wave solutions. Moreover, we provide some examples.

**Acknowledgements.** *The second named author is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. AP05131044).*

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*Received October 9, 2018*

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УДК 517.95

DOI: 10.14529/mmp190203

## О НЕЛИНЕЙНОЙ ЗАДАЧЕ ЛОМАЮЩИХСЯ ВОДНЫХ ВОЛН

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Настоящая работа посвящена начальной краевой задаче для уравнения Кортевега–де Фриза – Бенджамина – Бона – Махони в конечной области. Эта задача возникает из-за явления длинной волны с малой амплитудой в жидкости. Для некоторых начально-краевых задач для уравнения Кортевега–де Фриза – Бенджамина – Бона – Махони мы получили условия разрушения глобальных решений и решений типа бегущей волны за

конечное время. Доказательство результатов основано на методе нелинейной емкости. В заключение мы приводим точные и численные примеры.

*Ключевые слова:* разрушающие волны; уравнение Кортевега-де Фриза – Бенджамина – Бона – Махони; разрушение решение; начальные граничные задачи.

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*Поступила в редакцию 9 октября, 2018 г.*