ОБЗОРНЫЕ СТАТЬИ

MSC 35R30, 35Q35, 65M60, 65M32 **DOI: 10.14529/mmp210101**

ON EVOLUTIONARY INVERSE PROBLEMS FOR MATHEMATICAL MODELS OF HEAT AND MASS TRANSFER

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This article is a survey. The results on well-posedness of inverse problems for mathematical models of heat and mass transfer are presented. The unknowns are the coefficients of a system or the right-hand side (the source function). The overdetermination conditions are values of a solution of some manifolds or integrals of a solution with weight over the spatial domain. Two classes of mathematical models are considered. The former includes the Navier–Stokes system, the parabolic equations for the temperature of a fluid, and the parabolic system for concentrations of admixtures. The right-hand side of the system for concentrations is unknown and characterizes the volumetric density of sources of admixtures in a fluid. The unknown functions depend on time and some part of spacial variables and occur in the right-hand side of the parabolic system for concentrations. The latter class is just a parabolic system of equations, where the unknowns occur in the right-hand side and the system as coefficients. The well-posedness questions for these problems are examined, in particular, existence and uniqueness theorems as well as stability estimates for solutions are exposed.

Keywords: inverse problem; heat and mass transfer; filtration; diffusion; well-posedness.

Introduction

First, we consider the system

$$u_t - \nu\Delta u + (u,\nabla)u + \nabla p = f + \beta_c C + \beta_\theta \Theta, \quad \operatorname{div} u = 0, \tag{1}$$

$$\Theta_t - \operatorname{div}(\lambda_\theta \nabla\Theta) + (u,\nabla)\Theta = f_\theta, \tag{2}$$

$$C_t + (u,\nabla)C - Lu = f_c,\ Lu = \sum_{i,j=1}^{n} a_{ij} C_{x_i x_j} + \sum_{i=1}^{n} a_i C_{x_i} + a_0 C, \tag{3}$$

where $\nu = \text{const} > 0$, $(x,t) \in Q = G \times (0,T)$, $G \subset \mathbb{R}^n$, $T < \infty$, u, Θ, p, C are the velocity vector, the temperature of a fluid, the pressure, the concentrations of admixtures (inorganic or organic) in a fluid, and f_c is the volumetric density of sources of admixtures, respectively. The system (1) – (3) describes the propagation of admixtures in a fluid. In particular, it includes the classical Oberbeck–Boussinesq model (see, for instance, [1–3]). Here a_{ij}, a_i, a_0 are matrices of dimension $h \times h$, with h the number of admixtures, β_C is a matrix of dimension $n \times h$, β_Θ is a vector of length n, $\lambda_\Theta > 0$ is a scalar function. The description of these class of models can be found, for instance in [4], where even more general models can be found derived on the base of thermodynamics of noninvertible processes. The functions f_θ and f are the densities of the heat sources and external forces. The coefficient λ_θ stands for the thermal diffusivity. In the Oberbeck–Boussinesq model, the vector-functions β_c and β_θ are the mass transfer coefficient and the heat-transfer coefficient multiplied by the free fall acceleration. For generality, we assume below that β_c and β_θ are vector-functions of the variables (x,t).

For simplicity, we assume the domain G to be bounded though the main results are valid for a wide class of unbounded domains as well. The system (1) – (3) is furnished with the initial and boundary conditions

$$u|_{t=0} = u_0, \quad u|_S = g_1(t,x), \quad \Gamma = \partial G, \; S = \Gamma \times (0,T), \tag{4}$$

$$\Theta|_{t=0} = \Theta_0, \quad B_1\Theta|_S = g_2(t,x), \tag{5}$$

$$C|_{t=0} = C_0, \quad B_2 C|_S = g_3(t,x), \tag{6}$$

where $B_1 u = u$ or $B_1 u = \sum\limits_{i=1}^{n} \gamma_{1i}(x,t)\frac{\partial u}{\partial x_i} + \sigma_1(x,t)u$, with γ_{1i}, σ_1 some functions, and $B_2 u = u$ or $B_2 u = \sum\limits_{i=1}^{n} \gamma_{2i}(x,t)\frac{\partial u}{\partial x_i} + \sigma_2(x,t)u$, with γ_{2i}, σ_2 some matrices of dimension $h \times h$.

We consider an inverse problem of defining a solution to the system (1) – (3) and the right-hand side f_c in (3) using the data of additional measurements on cross-sections of G. Let $x'' = (x_{k+1}, x_{k+2}, \dots, x_n)$, $k = 0,1,\dots,n-1$. If $k \geq 1$ then we put $x' = (x_1, x_2, \dots, x_k)$. The right-hand side is of the form

$$f_c = f_0(x,t) + \sum_{i=1}^{r} f_i(x,t) q_i(x',t), \quad (x,t) \in Q, \tag{7}$$

where f_i $(i = 0,1,\dots,m)$ are given vector-functions. The functions $q_i(x',t)$ $(q_i(t))$ in this representation are unknown and the overdetermination conditions for recovering these functions are of the form

$$C|_{S_i} = \psi_i(t,x), \quad S_i = (0,T) \times \Gamma_i, \quad i = 1,2,\dots,s, \tag{8}$$

where $\{\Gamma_i\}$ is a collection of smooth k-dimensional surfaces lying in G_0. For $k = 0$, the surfaces Γ_i are just points lying in G. One more overdetermination condition is of the form

$$\int_G (C, \varphi_i(x))\,dx = \psi_i(t), \quad i = 1,2,\dots,r, \tag{9}$$

where the brackets denote the inner product in $\mathbb{R}^h$, $\varphi_i(x)$ is a vector with h components, and $\psi_i(t)$ are given functions. We do not know the articles where the inverse problems (1) – (8) or (1) – (7), (9) for the complete system are studied except for the author articles [5–7]. We can refer to [8], where a series of results devoted to optimal control problems for the systems occurring in the class (1) – (3) in the stationary case can be found. Optimal control problems for some simpler models are studied in [9–11]. The description of numerical methods of solving directs problems for Oberbeck–Boussinesq model is exposed in [3].

Many results connected with solvability of inverse problems for the Navier–Stokes system and the linearized Navier–Stokes system are presented in [12], where the main results are connected with the ovedetermination conditions of the form (9).

The inverse problems (1) – (8) and (1) – (7), (9) as well as the problems (3), (6), (8) and (3), (6), (9) for parabolic equations and systems arise when describing heat and mass transfer, filtration, diffusion, and some other physical processes [13, 14]. We can note that, in a real situation, even the simplest one-dimensional models used in monitoring and warning systems for river basins include several parabolic equations relative to concentrations. For parabolic equations and systems, the problems of the above type are studied in many articles and we can refer to the book [15], where these problems are discussed in the case of parabolic equations of the second order and $k = n - 1$. The overdetermination conditions here are values of a solution on sections of a spatial domain and the coefficients are independent of some spatial variables, the latter allows to apply the Fourier transform and to simplify the problem. The inverse problems with additional data on planes (sections of a spatial domain) are considered also in [16, 17] and some other articles. More general inverse problems with data on surfaces of arbitrary dimension are studied in [18–21]. The most known overdetermination conditions used in these problems are the values of a solution at some collection of interior points of G. Thus, additional conditions are the data of measurements (for example, the concentration of the transferred substance) at certain points in the domain. The data are employed to determine both the sources (for example, sources of pollution in water or air) and environmental parameters. The unknowns $q_i(x', t)$ depend in this case only on t. Thus, the right-hand side in (3) is representable as $f_c = f_0(x,t) + \sum\limits_{i=1}^{r} f_i(x,t)q_i(t)$. The inverse problem is to find a solution to the system (3) and the functions $q_i(t)$, $i = 1, 2, \ldots, r$, that appear in the right-hand side (3) or in the equation itself from the data (6) and (8), where $S_i = \{x_i\}$ are points. In the heat and mass transfer and filtration problems, the right-hand side f_c characterizes the distribution of sources (sinks) and their intensities. In the case of point sources, i.e. $f_i = \delta(x - x_i)$, where δ is the Dirac delta function, q_i is the intensity of the i-th source in the heat and mass transfer problems, and in filtration problems, for example, in oil production q_i is the flow rate i-th well, in this case u is the pressure [22]. In various practical problems distributed and point sources as well are both employed. First, we describe some results devoted to problems with spatially distributed sources. A large number of results was obtained in the case a linear second order equation.

We can refer to the article [23], where a theorem on the existence and uniqueness of solutions to problem (3), (6), (8) on determining the source in Hölder spaces in the case $h = 1$ and $r = 1$ is obtained. Similar results in the case of the problems of determining the source function and coefficients were obtained in the monograph [24] but in a one-dimensional situation ($n = 1$). In [25] the problem of determining the lowest order coefficient in a parabolic equation was considered, and in [26], the lowest order coefficient and the right-hand side of the form $q(t)f(t,x)$ are determined. In both cases, the well-posedness of the corresponding inverse problems is proven. There are many articles devoted to model equations and systems mainly in the one-dimensional situation (see, for example, [27, 28]). The first most essential results for quasilinear equations of the form (3) were obtained in [29], where conditions for a nonlinear function depending on $u, \nabla u$ were derived that guarantee the global solvability (in time) of the problem (3), (6), (8) in the Hölder spaces for case $r = 1$. The authors of [30] obtain a similar result already in the case of a parabolic system and in the Sobolev spaces. The problem of local or global well-posedness of linear and quasilinear problem of the form (3), (6), (8) in the Sobolev spaces was further considered in the articles [31–33]. In the general setting quasilinear inverse problems are considered in the book [12], where the relevant bibliography can be found. The authors consider a nonlinear nonautonomous first order operator-differential equation. The operator in the main part is a generator of an analytic semigroup. The overdetermination conditions are a collection of functionals defined on a given Banach space. The inverse parabolic problems with the overdetemination conditions (8), with $k = 0$, and (9) are thus included in this statement. The problem was studied in the spaces of functions continuously differentiable with respect to t. However, the constraints imposed on the nonlinearity are rather strong and can be essentially weakened. Weaker assumptions on the nonlinearity are used in [34], where the domain of the operator $A(t)$ in the main part can depend on time and the main results are stated in the Sobolev spaces. The article [35] contains the results on solvability of a linear inverse problem of recovering the function $f(t)$ in the operator-differential equation $u_t + Au = f(t)z$ with the overdermination conditions $\Phi(u) = \psi(t)$ (Φ is a functional). A huge amount of articles is devoted to numerical solving the problems of the form (3), (6), (8). We can refer, for example, to the articles [36–38]. There is a large number of monographs devoted to numerical methods for solving inverse problems. Almost all inverse parabolic problems and a large number of numerical methods are considered in the monograph [14] in the case $n = 1$. The monographs [39, 40] are devoted to a more general situation; number of interesting statements and problems (including those of convective heat transfer) are considered in [41, 42].

Describe some results in the case of point sources. As already noted, these problems are not well-posed in the classes of finite smoothness, and there are practically no existence and uniqueness theorems for solutions [43]. There is a huge number of articles devoted to the numerical solution of the problem of determining point sources, however, as a rule, these articles do not contain any theoretical justifications and very often both non-existence of solutions and their non-uniqueness can take place in the corresponding problems for certain values of parameters. The articles [45–47] can serve as examples. Let us single out the articles, where there is some theoretical justification of algorithms for finding solutions [48–53]. Note that in this case we need to determine the number of sources, their locations and intensities. The most interesting idea of constructing point sources is presented in [51]. It was subsequently used in [52]. Note that the problems of determining point sources are nonlinear, in contrast to the case of distributed sources.

The well-posedness questions for parabolic equations and systems with the overdetermination conditions (9) (including numerical methods) are treated in many articles. The first article probably is that by A.I. Prilepko [56] with coauthors, where the question of recovering the right-hand side $f = q(t)g(x,t) + f_0(x,t)$ (the unknown is the function $q(t)$) in a parabolic equation was examined in Hölder spaces. Next, we should refer to the well-known article [57] (see also [58]), where general nonlinear parabolic problems were considered in the one-dimensional case also in Hölder spaces. In particular, it is established under certain condition (at most linear growth of the nonlinearity in u, u_x at infinity if the main part of the equation is linear) that the solvability of the inverse problem with the overdetermination conditions of the form (9) is global in time. Next, we can refer to [59–67], where inverse problems of recovering coefficients depending on time in the case of the $r = 1$ in a linear parabolic equation. There are examples of simultaneous determination of the right-hand side and the coefficient (see, for instance, [68], where

$n = 1$). The inverse coefficient problems for the parabolic system of the second order having a special structure was studied in [59] in the Hölder spaces. Here the overdetermination conditions (9) and (8) for the parabolic system (3) are used simultaneously. Examples of recovering the right-hand sides in a second order parabolic equation are presented in [69–72]. The multidimensional inverse problems of recovering the right-hand side and coefficients simultaneously are studied also in [44, 73, 74] in the Sobolev spaces.

Among the monographs devoted to inverse problem for parabolic equation and systems we note the monographs [12, 14, 15, 42, 75–81], where the sufficient bibliography can be found.

Describe the contents of the article. The next section contains some conventional definitions. Section 2 contains the results on solvability of the problems (1) – (8) and (1) – (7), (9). Section 3 is devoted to solvability of the parabolic problems (3), (6), (8) and (3), (6), (9), where the operator L is replaced with a higher order elliptic operator. The notations of function spaces are conventional (see, for example, [54, 55]).

1. Definitions and Notations

Let E be a Banach space. By $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$) we mean the space of strongly measurable functions defined on G with values in E endowed with the norm $\|\|u(x)\|_E\|_{L_p(G)}$ [54]. We employ also the Hölder spaces $C^\alpha(\overline{G})$ (see the definition in [54]). The Sobolev space notations are conventional, i.e. $W_p^s(G;E)$, $W_p^s(Q;E)$, etc. (see the definitions in [54,55]) designate the Sobolev spaces of functions with values in E. If $E = \mathbb{C}$ ($E = \mathbb{R}$) or $E = \mathbb{C}^n$ ($E = \mathbb{R}^n$) then the latter space is denoted by $W_p^s(Q)$. Similarly, we use the notations $W_p^s(G)$ or $C^\alpha(\overline{G})$ rather than $W_p^s(G;E)$ or $C^\alpha(\overline{G};E)$. Thus, the membership $u \in W_p^s(G)$ (or $u \in C^\alpha(\overline{G})$) for a given vector-function $u = (u_1, u_2, \ldots, u_k)$ means that every of its component u_i belongs to $W_p^s(G)$ (or $C^\alpha(\overline{G})$). The norm of the vector is just the sum of the norms of the coordinates. The same meaning has the membership $a \in W_p^s(G)$ for a matrix-function a. Given an interval $J = (0,T)$, put $W_p^{s,r}(Q) = W_p^r(J; L_p(G)) \cap L_p(J; W_p^s(G))$. Respectively, $W_p^{s,r}(S) = W_p^r(J; L_p(\Gamma)) \cap L_p(J; W_p^s(\Gamma))$. Similarly, we can define the Hölder space $C^{r,s}(\overline{Q})$.

Next, we present some auxiliary statements. For simplicity, we assume that G is a bounded domain, though many of the results are valid in the case of unbounded domains as well. Let the symbol $B_\delta(x_i)$ stand for the ball centered at x_i of radius δ. As conventionally, we denote by $L_{p,\sigma}(G)$ the closure of solenoidal C_0^∞-vector-functions in the norm of $L_p(G)$ and put $W_{p,\sigma}^s(G) = W_p^s(G) \cap L_{p,\sigma}(G)$ and $W_{p,\sigma}^{s,s/2}(Q) = W_p^{s,s/2}(Q) \cap L_p(0,T; L_{p,\sigma}(G))$ ($s \geq 0$). The symbol $\overset{\circ}{W}{}_q^s(G)$ designates the closure of $C_0^\infty(G)$ in the norm of the space $W_q^s(G)$ and $\dot{W}_q^1(G) = \{p \in L_{q,loc}(G) : \ \nabla p \in L_q(G)\}$. We identify functions which differ by a constant and endow this space with the norm $\|p\|_{\dot{W}_q^1(G)} = \|\nabla p\|_{L_q(G)}$. It is a Banach space.

Consider the parabolic problem

$$u_t + Lu = f, \quad B_j u|_S = g_j, \quad u(0,x) = u_0(x), \tag{10}$$

where $Lu = \sum_{|\alpha| \leq 2m} a_\alpha(t,x) D^\alpha u$, $B_j u = \sum_{|\alpha| \leq m_j} b_{j\alpha}(t,x) D^\alpha u$. Introduce the operators $L_0 u = \sum_{|\alpha| = 2m} a_\alpha(t,x) D^\alpha u$ and $B_{0j} u = \sum_{|\alpha| = m_j} b_{j\alpha}(t,x) D^\alpha u$. We say that the problem (10) satisfies the (PL) condition (see [82, p. 198] and [83, Ch. 7]) if

(PL) a) there exists a constant $\delta_1 > 0$ such that any root p of the polynomial

$$\det\big(L_0(t,x,i\xi)+pE\big)=0$$

(E is the identity matrix) satisfies the inequality

$$\operatorname{Re} p \le -\delta_1|\xi|^{2m} \quad \forall \xi\in\mathbb{R}^n \quad \forall (x,t)\in Q;$$

b) for every point $(t_0,x_0)\in S$, $\xi\in\mathbb{R}^n$ such that $(\xi,\nu(x_0))=0$ ($\nu(x)$ is the outward unit normal to Γ at x), and all $\vec{h}\in\mathbb{C}^h$, λ such that $Re\,\lambda\ge 0$ and $|\xi|+|\lambda|\ne 0$, the system

$$\big(\lambda E+(-1)^m A_0(x_0,t_0,\xi+i\nu(x_0)\partial_y)\big)v(z)=0,\ y>0,\ \ B_0(x_0,t_0,\xi+i\nu(x_0)\partial_y)v(0)=h, \quad (11)$$

has a unique solution decreasing at infinity of the class $C([0,\infty))$.

The algebraic conditions ensuring the condition (PL) (the parabolicity condition and the Lopatinskii condition) can be found in [83], for example.

2. Inverse Problems (1) – (8) and (1) – (7), (9)

First, we describe the conditions on the data of the problems (1) – (8) and (1) – (7), (9).

(A) The case of $k>0$. There exists a domain $\Omega\subset\mathbb{R}^k$ with boundary of class C^2 such that $G\subset\Omega\times\mathbb{R}^{n-k}$,

$$\Gamma_i=\{x\in\mathbb{R}^n:\ x''=\varphi^i(x')=(\varphi^i_{k+1}(x'),\varphi^i_{k+2}(x'),\dots,\varphi^i_n(x')),x'\in\Omega\},$$

$\varphi^i(x')\in C^2(\overline{\Omega})$ and there exists a constant $\delta>0$ such that

$$U_{\delta i}=\{(x',\varphi^i(x')+\eta):\ x'\in\Omega,\ \eta\in\mathbb{R}^{n-k},\ |\eta|<\delta\}\subset G$$

for $i=1,2,\dots,s$, and $U_{\delta i}\cap U_{\delta j}=\emptyset$, for $i\ne j$, $i,j=1,2,\dots,s$.

The case of $k=0$. In this case the sets Γ_i are points and we assume that these points are interior points of G.

In what follows, we use the following notations: $Q_0=\Omega\times(0,T)$, $S_0=\partial\Omega\times(0,T)$, $Q_0^\tau=\Omega\times(0,\tau)$, $S^\tau=\Gamma\times(0,\tau)$, $G_\delta=\cup_i U_{\delta i}$, and $Q_\delta=G_\delta\times(0,T)$, $Q_\delta^\tau=G_\delta\times(0,\tau)$, $Q^\tau=G\times(0,\tau)$.

The condition (A) have been used in all articles devoted to the problems in question. As is easily seen, it ensures uniqueness of solutions. The condition (A) is fulfilled if $G=\Omega\times\mathbb{R}^{n-k}$, with Ω a bounded or unbounded domain of class C^2.

First, we present our conditions for the data. Let $\tilde{f}_j(x',x'',t)$ ($(x',t)\in\mathbb{R}^{k+1}$, $x''\in\mathbb{R}^{n-k}$, $j=1,2,\dots,r$) be the zero extension of the function f_j from Q to $\mathbb{R}^{n+1}$, i. e., $\tilde{f}_j\equiv 0$ on $\mathbb{R}^{n+1}\setminus Q$. In view of the condition (A), we can assume that the functions ψ_j ($j=1,2,\dots,s$) in (8) depends on the variables x',t only, i. e., $\psi_j=\psi_j(x',t)$ ($(x',t)\in Q_0$). We also assume that the parameter $\delta>0$ used below is that of this condition. First, we consider the case of the Dirichlet boundary conditions in Theorems 1, 2 below.

The agreement and smoothness conditions. Let $q>n+2$ and there exist vector-functions Φ_1, Φ_3 and function Φ_2 such that

$$\Phi_i(t,x)\in W_q^{2,1}(Q):\ \Phi_1|_{t=0}=u_0,\ \Phi_2|_{t=0}=\Theta_0,\ \Phi_3|_{t=0}=C_0,\ \Phi_i|_S=g_i, \quad (12)$$

$$\operatorname{div}\Phi_1 = 0,\ \Phi_3|_{S_j} = \psi_j,\quad f_0, f_\theta, f \in L_q(Q),\ \tilde{f}_j \in L_\infty(Q_0; L_q(\mathbb{R}^{n-s})), \tag{13}$$

$$\nabla_{x''}\Phi_3 \in W_q^{2,1}(Q_\delta),\quad \nabla_{x''} f_0 \in L_q(Q_\delta),\quad f_j, \nabla_{x''} f_j \in L_\infty(Q_\delta), \tag{14}$$

where $j = 1, 2, \ldots, r$ and $i = 1, 2, 3$. As a consequence of these conditions and embedding theorems, we can conclude that

$$u_0, C_0, \Theta_0 \in B_{q,q}^{2-2/q}(G),\ g_i \in W_q^{2-1/q, 1-1/(2q)}(S),\ i = 1, 2, 3,$$

$$\nabla_{x''} C_0 \in B_{q,q}^{2-2/q}(G_\delta),\ \nabla_{x''} g_3 \in W_q^{2-1/q, 1-1/(2q)}(S_\delta),\quad \psi_j(t, x') \in W_q^{2,1}(Q_0),$$

where $j = 1, 2\ldots, r$ and $S_\delta = (\partial G_\delta \cap \Gamma) \times (0, T)$. If these smoothness conditions and the corresponding agreement conditions (see trace theorems, for instance, in [54]) hold then we can construct the corresponding functions Φ_i. For example, if $g_1 = 0$, $\operatorname{div} u_0 = 0$, $q > 3/2$, and the above smoothness condition for u_0 holds then the agreement condition on the vector-function u_0 ensuring the existence of Φ_1 is the condition $u_0|_\Gamma = 0$.

Let $B(x', t)$ be the matrix whose rows with the numbers from $(j-1)h + 1$ to jh, $j = 1, 2, \ldots, s$ are occupied by the vectors

$$\left[f_1(x', \varphi^k(x'), t), f_2(x', \varphi^k(x'), t), \ldots, f_r(x', \varphi^k(x'), t)\right].$$

We require that there exist a constant $\delta_1 > 0$ such that

$$|\det\ B(x', t)| \ge \delta_1\ \text{ a.e. in } Q_0. \tag{15}$$

We also assume that

(B) $\lambda_\theta(x, t) \ge \delta > 0\ \forall (x, t) \in Q$, $\lambda_\theta(x, t) \in W_\infty^1(Q)$, $a_{ij} \in C(\overline{Q})$, and $\nabla_{x''} a_{ij} \in L_\infty(Q_\delta)$ for all $i, j = 1, 2, \ldots, n$; β_c, a_i, a_0, $\beta_\theta \in L_q(Q)$, $\nabla_{x''} a_i$, $\nabla_{x''} a_0 \in L_q(Q_\delta)$, $i = 1, 2, \ldots, n$. The proofs of Theorems 1, 2 below can be found in [5–7].

Theorem 1. *Assume that* $\Gamma \in C^2$, $q > n + 2$, *the problem* (3), (6) *satisfies the* (PL) *condition, and the conditions* (A), (B), (12) – (15) *hold. Then there exists a number* $\tau_0 \in (0, T]$ *such that there exists a unique solution* $(u, p, \Theta, C, q_1, \ldots, q_r)$ *to the problem* (1) – (8) *from the class*

$$u \in W_q^{2,1}(Q^{\tau_0}),\ p \in L_q(0, \tau_0; \dot{W}_q^1(G)),\quad q_j \in L_q(Q_0^{\tau_0}),\ j = 1, 2, \ldots, r,$$

$$\Theta, C \in W_q^{2,1}(Q^{\tau_0}),\quad \nabla_{x''} C \in W_q^{2,1}(Q_{\delta_2}^{\tau_0})\ \forall \delta_2 < \delta.$$

Let collections $(u^i, p^i, \Theta^i, C^i, q_1^i, \ldots, q_m^i)$, $i = 1, 2$ *be solutions to the problem* (1) – (8) *from the above class corresponding two different collections of the data* f^i, f_θ^i, f_0^i, ψ_j^i, u_0^i, g_η^i, Θ_0^i, *and* C_0^i, $j = 1, \ldots s$, $\eta = 1, 2, 3$, $i = 1, 2$ *satisfying* (12) – (14) *with some functions* Φ_j^i, $i = 1, 2$, $j = 1, 2, 3$ *and*

$$\sum_{j=1}^{3}\Big(\|\Phi_j^i\|_{W_q^{2,1}(Q)} + \|\nabla_{x''}\Phi_3^i\|_{W_q^{2,1}(Q_\delta)} + \|f^i\|_{L_q(Q)} +$$
$$+ \|f_\theta^i\|_{L_q(Q)} + \|f_0^i\|_{L_q(Q)} + \|\nabla_{x''} f_0^i\|_{L_q(Q_\delta)}\Big) \le R_0,\quad i = 1, 2.$$

Fix $\delta_2 < \delta$. Then there exists a constant $\tau_0 > 0$ such that the following stability estimate holds:

$$\|u^1 - u^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|\Theta^1 - \Theta^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|\nabla(p^1 - p^2)\|_{L_q(Q^{\tau_0})} +$$
$$+\|C^1 - C^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|\nabla_{x''}(C^1 - C^2)\|_{W_q^{2,1}(Q^{\tau_0}_{\delta_2})} + \sum_{j=1}^{r} \|a_j^1 - a_j^2\|_{L_q(Q_0^{\tau_0})} \leq$$
$$\leq c\big(\sum_{j=1}^{3} \|\Phi_j^1 - \Phi_j^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|\nabla_{x''}(\Phi_3^1 - \Phi_3^2)\|_{W_q^{2,1}(Q^{\tau_0}_{\delta})} + \|f_0^1 - f_0^2\|_{L_q(Q^{\tau_0})} +$$
$$+\|f^1 - f^2\|_{L_q(Q^{\tau_0})} + \|f_\theta^1 - f_\theta^2\|_{L_q(Q^{\tau_0})} + \|\nabla_{x''}(f_0^1 - f_0^2)\|_{L_q(Q^{\tau_0}_{\delta})}\big),$$

where the constant c depends on the quantities R_0 and δ_2.

Proceed with the linearized statement. We examine the system

$$u_t - \nu\Delta u + \nabla p = \sum_{j=1}^{n} B_j u_{x_j} + B_0 u + f + \beta_c C + \beta_\theta \Theta, \quad \operatorname{div} u = 0, \tag{16}$$

$$\Theta_t - \lambda_\theta \Delta\Theta + \sum_{j=1}^{n} b_j \Theta_{x_j} + b_0\Theta = f_\theta + \sum_{j=1}^{n} b^j u_j, \tag{17}$$

$$C_t - Lu = f_c + \sum_{j=1}^{n} c^j u_j, \; Lu = \sum_{i,j=1}^{n} a_{ij} C_{x_i x_j} + \sum_{i=1}^{n} a_i C_{x_i} + a_0 C, \tag{18}$$

where B_j, B_0 are $n \times n$ matrices. We assume that

(C) $b_j(x,t), b_0, b^j, c_j, c^j, c_0, B_j, B_0 \in L_q(Q), \quad \nabla_{x''} c^j(x,t) \in L_q(Q_\delta),$

where $j = 1, 2, \ldots, n$ and δ is the parameter that of the condition (A).

Theorem 2. *Assume that $\Gamma \in C^2$, $q > n + 2$, the problem* (3), (6) *satisfies the* (PL) *condition, and the conditions* (A), (B), (C), (12) – (15) *hold. Then there exists a unique solution $(u, p, \Theta, C, q_1, \ldots, q_r)$ to the problem* (16) – (18), (4) – (8) *from the class*

$$u \in W_q^{2,1}(Q), \; p \in L_q(0,T; \dot{W}_q^1(G)), \quad q_j \in L_q(Q_0), \; j = 1, 2, \ldots, r,$$

$$\Theta, C \in W_q^{2,1}(Q), \quad \nabla_{x''} C \in W_q^{2,1}(Q_{\delta_2}) \; \forall \delta_2 < \delta.$$

Fix $\delta_2 < \delta$. A solution meets the estimate

$$\begin{gathered}\|u\|_{W_q^{2,1}(Q)} + \|\Theta\|_{W_q^{2,1}(Q)} + \|\nabla p\|_{L_q(Q)} + \|C\|_{W_q^{2,1}(Q)} + \sum_{j=1}^{r} \|q_j\|_{L_q(Q_0)} + \\ +\|\nabla_{x''} C\|_{W_q^{2,1}(Q_{\delta_2})} \leq c(\sum_{i=1}^{3} \|\Phi_i\|_{W_q^{2,1}(Q)} + \|\nabla_{x''}\Phi_3\|_{W_q^{2,1}(Q_\delta)} + \\ +\|f_0\|_{L_q(Q)} + \|\nabla_{x''} f_0\|_{L_q(Q_\delta)} + \|f\|_{L_q(Q)} + \|f_\theta\|_{L_q(Q)}).\end{gathered} \tag{19}$$

Next, we consider the integral overdetermination conditions (9). Actually, these results are new. We describe them without proofs. The proofs can be found in the forthcoming paper in Itogi Nauki i Tekhniki (2020, vol. 187).

In this case, we have that $q_i = q_i(t)$, i. e., the functions q_i depend only on t.

Our conditions on the data can be written in the following form

$$u_0 \in W_q^{2-2/q}(G),\ div\, u_0 = 0,\ u_0|_\Gamma = 0,\ f, f_\theta, f_0 \in L_q(Q),\ q > n+2, \tag{20}$$

$$C_0(x) \in W_q^{2-2/q}(G),\quad g_3(x,t) \in W_q^{s_2,2s_2}(S),\ B_2(x,0,D)C_0|_\Gamma = g_3(x,0), \tag{21}$$

$$\Theta_0(x) \in W_q^{2-2/q}(G),\quad g_2(x,t) \in W_q^{s_1,2s_1}(S),\ B_1(x,0,D)\Theta_0|_\Gamma = g_2(x,0), \tag{22}$$

where $s_i = 1 - 1/2q$ if $B_i u = u$ $(i = 1,2)$ and $s_i = 1/2 - 1/2q$ otherwise and we take $g_1(x,t) \equiv 0$ in (4)

$$\psi_i(t) \in W_q^1(0,T),\ \psi_i(0) = \int\limits_{G_i} (C_0(x), \varphi_i(x))\, dx,\ i = 1,2,\ldots,r, \tag{23}$$

$$\begin{gathered} a_i(t,x) \in L_q(Q)\ (i = 0,1,\ldots,n),\quad a_{ij} \in C([0,T]; C^{\varepsilon_0}(\overline{G})),\ i,j = 1,\ldots,n, \\ \gamma_{ij}, \sigma \in C^{1/2-1/2p+\varepsilon_0, 1-1/p+\varepsilon_0}(\overline{S}),\ j = 1,\ldots,n,\ i = 1,2, \end{gathered} \tag{24}$$

where $\varepsilon_0 > 0$ is a positive constant

$$f_i(x,t) \in L_\infty(0,T; L_q(G)),\ i = 1,2,\ldots,r. \tag{25}$$

Let $\{G_j\}$ be a collection of subdomains of G with boundaries of the class C^1. We assume that

$$\varphi_j \in L_p(G),\ \mathrm{supp}\varphi_j \subset G_j \subset G,\ \varphi_j \in W_p^{\varepsilon_1}(G_j),\ \tfrac{1}{q} + \tfrac{1}{p} = 1,\ j = 1,2,\ldots,r, \tag{26}$$

for some $\varepsilon_1 > 0$.

Define the entries $b_{ij}(t)$ of the matrix B by the equalities $b_{ij} = \int\limits_G (f_j, \varphi_i(x))\, dx$ and suppose that there exist constants $\delta_0, \delta_1 > 0$ such that

$$|\det B| \geq \delta_0 > 0,\ \text{a.a. on } (0,T), \tag{27}$$

$$\lambda_\theta(x,t) \geq \delta_1 > 0,\ (x,t) \in Q), \lambda_\theta \in C(\overline{Q});\ \beta_c, \beta_\theta \in L_q(Q). \tag{28}$$

Introduce the set B_R of vectors $\vec{U} = (u_0, C_0, \Theta_0, g_2, g_3, f, f_0, f_\theta, \psi_1, \ldots, \psi_r)$, satisfying (20) – (23) and such that

$$\begin{gathered} \|u_0\|_{W_q^{2-2/q}(G)} + \|C_0\|_{W_q^{2-2/q}(G)} + \|\Theta_0\|_{W_q^{2-2/q}(G)} + \|g_2\|_{W_q^{s_1,2s_1}(S)} + \|g_3\|_{W_q^{s_2,2s_2}(S)} + \\ + \|f\|_{L_q(Q)} + \|f_\theta\|_{L_q(Q)} + \|f_0\|_{L_q(Q)} + \sum_{i=1}^{s} \|\psi_i\|_{W_q^1(0,T)} \leq R. \end{gathered}$$

Theorem 3. *Assume that $\Gamma \in C^2$, the problems* (3), (6) *and* (2), (5) *satisfy the* (PL) *condition, $q > n+2$, and the condition* (20) – (28) *hold. Then there exists a number $\tau_0 \in (0,T]$ such that there exists a unique solution $(u, p, \Theta, C, q_1, \ldots, q_r)$ to the problem* (1) – (7), (9) *from the class*

$$u \in W_q^{2,1}(Q^{\tau_0}),\ p \in L_q(0,\tau_0; \dot{W}_q^1(G)),\quad q_j \in L_q(Q_0^{\tau_0}),\ j = 1,2,\ldots,r.$$

Fix $R_0 > 0$. Then there exist constants $\tau_0 = \tau_0(R_0)$ and $c = c(R_0)$ such that for every two solutions u^i, Θ^i, C^i, q^i, $q^i = (q_{i1}, q_{i2}, \ldots, q_{ir})$, $i = 1, 2$ relating to the collections $\vec{U}_1, \vec{U}_2 \in B_{R_0}$, $\vec{U}_i = (u_0^i, C_0^i, \Theta_0^i, g_2^i, g_3^i, f^i, f_0^i, f_\theta^i, \psi_1^i, \ldots, \psi_r^i)$, $i = 1, 2$, the following estimate holds:

$$\|u^1 - u^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|\Theta^1 - \Theta^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|C^1 - C^2\|_{W_q^{2,1}(Q^{\tau_0})} + \\ + \sum_{j=1}^{r} \|q_{1j} - q_{2j}\|_{L_q(0,\tau_0)} \le c\Big(\|u_0^1 - u_0^2\|_{W_q^{2-2/q}(G)} + \|f^1 - f^2\|_{L_q(Q^{\tau_0})} + \\ + \|f_\theta^1 - f_\theta^2\|_{L_q(Q^{\tau_0})} + \|f_0^1 - f_0^2\|_{L_q(Q^{\tau_0})} + \|C_0^1 - C_0^2\|_{W_q^{2-2/q}(G)} + \|\Theta_0^1 - \Theta_0^2\|_{W_q^{2-2/q}(G)} + \\ + \|g_2^1 - g_2^2\|_{W_q^{s_1,2s_1}(S^{\tau_0})} + \|g_3^1 - g_3^2\|_{W_q^{s_2,2s_2}(S^{\tau_0})} + \sum_{i=1}^{s} \|\psi_i^1 - \psi_i^2\|_{W_q^1(0,\tau_0)}\Big).$$

Consider a linearized statement. We consider the system (16) – (18), where

$$B_0, b^j, b_0, a^j, B_j, b_j, \in L_q(Q),\ j = 1, 2, \ldots, n. \tag{29}$$

Theorem 4. *Assume that $\Gamma \in C^2$, $p > n + 2$, the problems* (3), (6) *and* (2), (5) *satisfy the* (PL) *condition, and the conditions* (20) – (29) *hold. Then there exists a unique solution $(u, p, \Theta, C, q_1, \ldots, q_r)$ to the problem* (16) – (18), (4) – (7), (9) *from the class*

$$u \in W_q^{2,1}(Q),\ p \in L_q(0, T; \dot{W}_q^1(G)),\quad q_j \in L_q(Q),\ j = 1, 2, \ldots, r.$$

A solution satisfies the estimate

$$\|u\|_{W_q^{2,1}(Q)} + \|\Theta\|_{W_q^{2,1}(Q)} + \|C\|_{W_q^{2,1}(Q)} + \sum_{j=1}^{r} \|q_j\|_{L_q(0,T)} \le c\Big(\|u_0\|_{W_q^{2-2/q}(G)} + \|f\|_{L_q(Q)} + \\ + \|f_\theta\|_{L_q(Q)} + \|f_0\|_{L_q(Q)} + \|C_0\|_{W_q^{2-2/q}(G)} + \|\Theta_0\|_{W_q^{2-2/q}(G)} + \\ + \|g_2\|_{W_q^{1-1/2q,2-1/q}(S)} + \|g_3\|_{W_q^{s_0,2s_0}(S)} + \sum_{i=1}^{r} \|\psi_i\|_{W_q^1(0,T)}\Big).$$

3. Inverse Problems for Parabolic Systems

In this section, we examine parabolic equations and systems of the form

$$u_t + A(t, x, D)u = \sum_{i=1}^{r} b_i(t, x) q_i(t, x') + f,\quad (t, x) \in Q,\quad x = (x', x''), \tag{30}$$

where $x' = (x_1, x_2, \ldots, x_k)$, $x'' = (x_{k+1}, x_{k+2}, \ldots, x_n)$, b_i, $i = 1, 2, \ldots, r$, and f are given vector-functions and A is a matrix elliptic operator of order $2m$ with matrix coefficients of dimension $h \times h$ representable as

$$A(t, x, D) = \sum_{i=r+1}^{r_0} q_i(t, x') A_i(t, x, D_x) + A_{r_0+1}(t, x, D_x), \tag{31}$$

$$A_i = \sum_{|\alpha| \le 2m} a_{i\alpha}(t, x) D^\alpha,\quad i = r + 1, \ldots, r_0 + 1,\ r_0 = sh,\ D = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}). \tag{32}$$

The unknowns in (30) are a solution u and functions $q_i(t, x')$, $i = 1, 2, \ldots, r_0$ occurring in the right-hand side (30) and the operator A as well; in the latter case they are coefficients. The equation (30) is complemented with the initial and boundary conditions

$$u\big|_{t=0} = u_0, \quad B_j u\big|_S = \sum_{|\beta|\le m_j} b_{j\beta}(t,x) D^\beta u\big|_S = g_j(t,x), \tag{33}$$

where $m_j < 2m$, $j = 1, 2, \ldots, m$. Let $\Gamma \in C^{2m}$. The overdetermination conditions have the form

$$u|_{S_i} = \psi_i(t,x), \quad i = 1, 2, \ldots, s. \tag{34}$$

The results exposed in this section can be found in author's articles [18–21, 44]. The condition on the data are written as follows.

$$\begin{gathered} \exists \Phi(t,x) \in W_p^{2m,1}(Q),\ p > n+2m:\ \Phi\big|_{t=0} = u_0(x), \quad B_l\Phi\big|_S = g_l,\ l = 1,\ldots,m, \\ \partial_{x_i}\Phi \in W_p^{2m,1}(Q_\delta),\ \Phi\big|_{S_j} = \psi_j(t,x') \in C\big([0,T]; C^{2m}(\Omega)\big),\ \psi_{jt} \in C\big(\overline{Q_0}\,\big), \\ f \in L_p(Q), \quad \partial_{x_i} f \in L_p(Q_\delta), \quad f\big|_{S_j} \in C\big(\overline{Q_0}\,\big),\ i \ge k+1,\ j = 1,\ldots,s. \end{gathered} \tag{35}$$

As a consequence of the conditions (35) and the embedding theorems, we have

$$u_0(x) \in W_p^{2m-2m/p}(G),\ g_j \in W_p^{2mk_j,k_j}(S),\ B_j u_0|_\Gamma = g_j(0,x), \tag{36}$$

where $k_j = (2m - m_j - 1/p)/(2m)$ and $j = 1, 2, \ldots, m$,

$$\partial_{x_i} g_j \in W_p^{2mk_j,k_j}(S_\delta), \quad \partial_{x_i} u_0(x) \in W_p^{2m-2m/p}(G_\delta),\ j = 1,2,\ldots,m, \quad i = k+1,\ldots,n, \tag{37}$$

where $S_\delta = (\partial G_\delta \cap \Gamma) \times (0,T)$. The conditions on the coefficients of the operators A, B_j are more or less conventional. For simplicity, we will use the conditions that are not sharp. We assume that

$$a_{i\alpha}(t,x) \in L_\infty(Q)\ \big(|\alpha| < 2m\big),\ a_{i\alpha} \in C\big(\overline{Q}\,\big)\ (|\alpha| = 2m),\ i = r+1,\ldots,r_0+1, \tag{38}$$

$$b_{j\beta} \in C^{2m-m_j,1-\frac{m_j}{2m}}(\overline{S}),\ \partial_{x_i} b_{j\beta} \in C^{2m-m_j,1-\frac{m_j}{2m}}(\overline{S_\delta}), \quad |\beta| \le m_j, j = 1,\ldots,m,\ i > k, \tag{39}$$

$$\partial_{x_i} a_{j\alpha}(t,x) \in L_\infty(Q_\delta),\ |\alpha| \le 2m, \quad j = r+1,\ldots,r_0+1,\ i > k, \tag{40}$$

$$b_l(t,x) \in L_p(Q),\ \partial_{x_i} b_l \in L_p(Q_\delta),\ (l = 1,\ldots,r,\ i > k). \tag{41}$$

We look for the functions q_i in the class of continuous functions. Hence, we require that

$$a_{i\alpha}\big(t,x',\varphi^j(x')\big),\ b_l\big(t,x',\varphi^j(x')\big) \in C\big(\overline{Q_0}\,\big) \tag{42}$$

for all $l = 1,\ldots,r$, $j = 1,2,\ldots,s$, and $|\alpha| < 2m$.

Now we introduce the matrix $B(t,x')$ of dimension $sh \times sh$ whose rows with the numbers from $(j-1)h+1$ to jh are occupied by the column vectors

$$\big(-b_1(t,x), -b_2(t,x), \ldots, -b_r(t,x), A_{r+1}\Phi(t,x), \ldots, A_{sh}\Phi(t,x)\big)\Big|_{x''=\varphi^j(x')}.$$

It can be shown with the use of the embedding theorems and the conditions (35), (36), (42) that the elements of this matrix are continuous on $\overline{G}$. We require also that there exists a constant $\delta_0 > 0$ such that

$$\big|\det B(t,x')\big| \ge \delta_0 \quad \forall x' \in \Omega, \quad t \in [0,T]. \tag{43}$$

Consider the system of equations

$$B(t,x')\vec{q}^{\,0}=\vec{g},\quad \vec{q}^{\,0}=\big(q_1^0,q_2^0,\dots,q_{sh}^0\big), \tag{44}$$

where $\vec{g}$ is the column vector whose coordinates with the numbers from $(j-1)h+1$ to jh are the vector

$$f\big(t,x',\varphi^j(x')\big)-A_{sh+1}\Phi\big(t,x',\varphi^j(x')\big)-\Phi_t\big(t,x',\varphi^j(x')\big). \tag{45}$$

Under the condition (43) the system (44) has a unique solution $\vec{q}^{\,0}=(q_1^0,\dots,q_{sh}^0)=\big(B(t,x')\big)^{-1}\vec{g}(t,x')$. The above conditions for the data ensure that $\vec{q}^{\,0}\in C\big(\overline{Q_0}\big)$. Consider the operator

$$A_0(t,x,D)=\sum_{i=r+1}^{sh}q_i^0(t,x')A_i(t,x,D_x)+A_{hs+1}(t,x,D_x),$$

and the problem

$$u_t+A_0(t,x,D_x)u=g\ \ \big((t,x)\in Q\big),\ \ u\big|_{t=0}=u_0(x),\ \ B_ju\big|_S=g_j. \tag{46}$$

Fix $i\in\{1,2,\dots,s\}$ and make the change of the variables $y'=x'$, $y''=x''-\varphi^i(x')$, $t=t$ in the domain $Q_{\delta_1 i}$, with $\delta_1<\delta$. After this change, the operators A and B_j are transformed into some operators $A^i(t,y,D_y)$ and $B_j^i(t,y,D_y)$. Denote by $A^i_{y'}$ and $B^i_{jy'}$ the parts of the operators A^i and B^i_j not containing the derivatives with respect to the variables $y_{k+1},y_{k+2},\dots,y_n$ and by $A^i_{y''}$ and $B^i_{jy''}$ the remainders. Similar sense has the notations $A_{x'}$, $B_{jx'}$, $A_{x''}$, $B_{jx''}$, and $A_{0x'}$, $A_{0x''}$. Describe the connections between the derivatives with respect to the new and old variables. We have

$$\partial_{x_j}=\partial_{y_j}-\sum_{r=k+1}^{n}\varphi^i_{ry_j}(y')\partial_{y_r},\ \ j\le k,\ \ \partial_{x_j}=\partial_{y_j},\ \ j>k,$$

$$\partial_{y_j}=\partial_{x_j}+\sum_{r=k+1}^{n}\varphi^i_{rx_j}(x')\partial_{x_r},\ \ j\le k,\ \ \partial_{y_j}=\partial_{x_j},\ \ j>k.$$

Thus, we infer

$$A^i_{y'}(t,y,D_{y'})=A_{x'}\big(t,y',y''+\varphi^i(y'),D_{y'}\big),\ B^i_{jy'}(t,y,D_{y'})\ =B_{jx'}\big(t,y',y''+\varphi^i(y'),D_{y'}\big).$$

As is easily seen, the operators $A_{x'}$ and preserve their form. Consider the auxiliary problems

$$\psi^j_t+A^j_{0y'}(t,y',0,D_{y'})\psi^j=0,\ \ (t,y')\in Q_0, \tag{47}$$

$$\psi^j(0,y')=0,\ j=1,2,\dots,s, \tag{48}$$

$$B^j_{iy'}\psi^j\big|_{S_0}=0,\ \ j=1,2,\dots,s,\ \ i=1,2,\dots,m. \tag{49}$$

Theorem 5. *Assume that the condition* (A)*, where* $\partial\Omega\in C^{2m}$ *and the conditions* (35), (38) – (43) *are fulfilled and the problem* (46) *satisfies the condition* (PL)*. If* $B^j_{iy''}(t,y',0,D_y)=0$ *for all* $i=1,2,\dots,m$ *and* $j=1,2,\dots,s$ *then there exists a number* $\tau_0\in(0,T]$ *such that there exists a unique solution* $(u,q_1,\dots,q_{sh})$ *to the problem* (30), (33), (34) *of the class*

$$u\in W_p^{2m,1}(Q^{\tau_0}):\nabla_{x''}u\in W_p^{2m,1}(Q^{\tau_0}_{\delta_2})\ \ \forall\delta_2<\delta,\ \ q_j\in C\big(\overline{Q^{\tau_0}_0}\big),\ j=1,2,\dots,sh.$$

Proceed with the linear case. Assume that all coefficients of the operator A are known functions, i.e., $r = sh$, $A = \sum\limits_{|\alpha|\le 2m} a_\alpha(t,x)D^\alpha u$, and all unknown functions q_i enter the right-hand side of (30). All conditions for the data have the same form. In particular, we assume in the next theorem that the problem $u_t + Au = f$, $u(0,x) = u_0$, $B_j u|_S = g_j$, $j = 1,\dots,m$ satisfies the confition (PL). In our case the rows of the matrix $B(t,x')$ of dimension $sh \times sh$ with the numbers from $(j-1)h+1$ to jh are occupied by the column vectors $\big(-b_1\big(t,x',\varphi^j(x')\big), -b_2\big(t,x',\varphi^j(x')\big),\dots, -b_r(t,x',\varphi^j(x')\big)$. The conditions (35), (41) can be rewritten as follows:

$$\exists\Phi(t,x)\in W_p^{2m,1}(Q)\ (p>n+2m):\ \Phi\big|_{t=0} = u_0(x),\quad B_l\Phi\big|_S = g_l,\ l=1,\dots,m,$$
$$\partial_{x_i}\Phi\in W_p^{2m,1}(Q_\delta), f\in L_p(Q),\quad \partial_{x_i} f\in L_p(Q_\delta),\ i\ge k+1, \quad (50)$$

$$b_l(t,x)\in L_\infty(Q),\ \partial_{x_i} b_l\in L_\infty(Q_\delta),\ l=1,\dots,r,\ i\ge k+1. \quad (51)$$

Let Ψ_0 be the class of vector-functions $\vec\psi = (\psi^1,\psi^2,\dots,\psi^s) \in W_p^{2m,1}(Q_0)$ whose coordinates meet (47), (48) and there exists a function Φ satisfying (50), with $u_0 = 0, g_j \equiv 0$ $(j=1,\dots,m)$, such that $B_{ix'}\big(t,x',\varphi^j(x'),D_{x'}\big)\psi^j\big|_{S_0} = B_{ix'}\big(t,x',\varphi^j(x'),D_{x'}\big)\Phi\big|_{S_0}$, where $i = 1,2,\dots,m$, $j=1,\dots,s$. We say that the equalities (34) *are fulfilled in a generalized sense* whenever there exists a vector-function $\vec\psi = (\psi^1,\psi^2,\dots,\psi^s)\in\Psi_0$ such that

$$u\big|_{S_i} = \psi_i(t,x') + \psi^i(t,x'),\quad (t,x')\in Q_0,\quad i=1,2,\dots,s. \quad (52)$$

The fulfillment of the equality (34) in a generalized sense means that it is fulfilled in the quotient space $\big(W_p^{2m,1}(Q_0)\big)^s/\Psi_0$, where $W_p^{2m,1}(Q_0)$ is a space of vector-functions $\vec\psi\in W_p^{2m,1}(Q_0)$ of length h.

Theorem 6. *Assume that the condition* (A)*, where* $\partial\Omega\in C^{2m}$*, and the conditions* (38)–(40), (43), (50), (51) *are fulfilled. Fix* $\delta_1<\delta$*. Then the following statements are valid.*

1. *There exists a constant* $c>0$ *such that a solution* $(u,q_1,\dots,q_r)$ *to the problem* (30) – (34) *from the class*

$$u\in W_p^{2m,1}(Q):\nabla_{x''}u\in W_p^{2m,1}(Q_{\delta_2})\quad\forall\delta_2<\delta,\quad q_j\in L_p(Q_0),\quad j=1,2,\dots,sh$$

meets the estimate

$$\|u\|_{W_p^{2m,1}(Q)} + \|\nabla_{x''}u\|_{W_p^{2m,1}(Q_{\delta_1})} + \sum_{j=1}^r \|q_j\|_{L_p(Q_0)} \le$$
$$\le c(\|\Phi\|_{W_p^{2m,1}(Q)} + \|\nabla_{x''}\Phi\|_{W_p^{2m,1}(Q_\delta)} + \|f\|_{L_p(Q)} + \|\nabla_{x''}f\|_{L_p(Q_\delta)}). \quad (53)$$

2. *There exists a unique solution* $(u,q_1,\dots,q_r)$ *to the problem* (30) – (34)*, where* (34) *is understood in the generalized sense, from the class*

$$u\in W_p^{2m,1}(Q),\quad \nabla_{x''}u\in W_p^{2m,1}(Q_{\delta_1})\quad\forall\delta_1<\delta,\quad q_j\in L_p(Q_0),\quad j=1,2,\dots,r.$$

3. *Solutions* $(u,q_1,\dots,q_r)$ *to the problem* (30) – (34)*, with* $u_0\equiv 0$*,* $f\equiv 0$*,* $g_j\equiv 0$ *and* $\psi = (\psi_1,\psi_2,\dots,\psi_s)\in\Psi_0$*, from the class*

$$u\in W_p^{2m,1}(Q):\nabla_{x''}u\in W_p^{2m,1}(Q_{\delta_1})\quad\forall\delta_1<\delta,$$

do not exist whenever $\psi\not\equiv 0$.

4. *If* $B^j_{iy''}(t,y',0,D_y)=0$ *for all* $i=1,2,\dots,m$ *and* $j=1,2,\dots,s$ *then* $\Psi_0=\{0\}$ *and there exists a unique solution* $(u,q_1,\dots,q_s)$ *to the problem* (30) – (34), *where the equality* (34) *is understood in the usual sense from the class*

$$u\in W_p^{2m,1}(Q):\nabla_{x''}u\in W_p^{2m,1}(Q_{\delta_1})\ \ \forall\delta_1<\delta,\ \ q_j\in L_p(Q_0),\ \ j=1,2,\dots,r.$$

Remark 1. We note that the function q_i are sought in the space $L_p(Q_0)$ in the previous theorem. However, the results are valid if a solution is sought in the class indicated in Theorem 5. The conditions on the data and the coefficients in this case coincide with those of Theorem 5 (see [19]).

Remark 2. Note that the above theorems are valid in the case of the pointwise overdetermination as well, i.e., $k=0$. The condition (A) in this case is reduced to the following conditions: the points $\{x_i\}_{i=1}^s$ are interior points of G. Moreover, in this case we can replace the conditions (35) with the more natural conditions (36), (37) and the consistency condition $u_0|_{S_l}=\psi_l(0,x')$, $l=1,\dots,s$.

Remark 3. If the condition (43) fails then very often the problem becomes ill-posed in the Hadamard sense. In this case the problem becomes unsolvable if the data have finite smoothness. The corresponding example can be found in [7, Example 3]. The condition of additional smoothness of the data in some neighborhood G_δ about the set, where the overdetermination data are imposed (see conditions (35), (37), (41), (40), etc.) also cannot be omitted. For example, in the condition (35) we require that $\nabla_{x''}f\in L_p(Q_\delta)$). In the case of the pointwise overdetermination this condition can be written as $f\in L_p(0,T;W_p^1(G_\delta))$ (G_δ is a neighborhood of the set $\{x_i\}$ of the overdetermination points). We can replace this condition with the condition $f\in L_p(0,T;W_p^s(G_\delta))$ with $s>n/p$ (see [31]). But if $s\le n/p$ we can construct ill-posedness examples again. In the general case of $k>0$ additional smoothness in the variables x'' can be characterized by the number $s>(n-k)/p$.

Next, we present an analog of Theorem 3 in the case of a higher order parabolic system. The results are published in [44, 73, 74]. We consider the problem (30), (33), where the operator A admits the representation (31). We slightly refine some of the statements in these articles. We assume that

$$\varphi_i\in L_q(G),\ \mathrm{supp}\varphi_i\subset G_i\subset G,\ \partial G_i\subset C^1,\ \varphi_i\in W_q^1(G_i),\ i=1,\dots,r_0,\ \frac{1}{p}+\frac{1}{q}=1,\ \ (54)$$

$$f\in L_p(Q),\ p>n+2m,\ b_i(x,t)\in L_\infty(0,T;L_p(G)),\ i=1,2,\dots,r,\qquad (55)$$

$$\psi_i(t)\in W_p^1[0,T],\ \psi_i(0)=\int\limits_{G_i}(u_0(x),\varphi_i(x))dx,\ i=1,2,\dots,r_0,\qquad (56)$$

$$a_{i\alpha}\in C(\overline{Q}),\ |\alpha|=2m,\ a_{i\alpha}\in L_p(Q),\ |\alpha|<2m,\ b_{j\beta}\in C^{2m-m_j,1-\frac{m_j}{2m}}(\overline{S}),\qquad (57)$$

where $j=1,\dots,m$, $|\beta|\le m_j$. Let $G_0=\cup_{i=1}^r G_i$ and assume that

$$b_j,f\in C([0,T];L_p(G_0))\ (j=1,2,\dots,r),\ a_{i\alpha}\in C([0,T],W_p^1(G_0))\ \text{for}\ |\alpha|=2m,\qquad (58)$$

$$a_{i\alpha}\in C([0,T],L_p(G_0)),\ \ i=r+1,r+2,\dots,r_0+1\ \ \text{for}\ |\alpha|<2m.\qquad (59)$$

Define the matrix B of dimension $r_0 \times r_0$ with the rows

$$\int\limits_G (b_1(0,x), \varphi_k)\, dx, ..., \int\limits_G (b_r(0,x), \varphi_k)\, dx,$$
$$-\int\limits_G (A_{r+1}(0,x)u_0, \varphi_k)\, dx, \ldots, -\int\limits_G (A_{r_0}(0,x)u_0, \varphi_k)\, dx.$$

We require that

$$\det\, B \neq 0. \tag{60}$$

Determine the constants q_i^0, $i = 1, 2, \ldots, r_0$ as solutions to the system

$$\begin{aligned} \psi_{jt}(0) + \sum_{i=r+1}^{r_0} q_i^0 \int\limits_G (A_i u_0, \varphi_j)\, dx + \int\limits_G (A_{r_0+1} u_0, \varphi_j)\, dx = \\ = \sum_{i=1}^{r} q_i^0 \int\limits_G (b_i(0,x), \varphi_j)\, dx + \int\limits_G (f, \varphi_j)\, dx, \end{aligned} \tag{61}$$

where $j = 1, 2, \ldots, r_0$, and construct the operator $A_0 = \sum\limits_{i=r+1}^{r_0} q_i^0 A_i + A_{r+1}$. Our overdetermination conditions take the form

$$\int\limits_G (u, \varphi_i(x))\, dx = \psi_i(t), \quad i = 1, 2, \ldots, r_0. \tag{62}$$

Theorem 7. *Assume that the conditions* (36), (54) – (60) *hold and the problem* (46) *satisfies the condition* (PL) *with the above-defined operator* A_0. *Then there exists a number* $\tau_0 \leq T$ *such that on the segment* $[0, \tau_0]$ *there exists a unique solution* $(u, q_1, .., q_{r_0})$ *to the problem* (30), (33), (62) *such that*

$$u \in W_p^{2m,1}(Q^{\tau_0}),\ q_i(t) \in C([0, \tau_0]),\ i = 1, 2, ..., r_0.$$

In the linear case, we can weaken our conditions on the coefficients. In this case all coefficients of the operator $A = A_0 = \sum\limits_{|\alpha| \leq 2m} a_\alpha D^\alpha$ are known functions and the unknowns q_i enter the right-hand side of (30). We require that

$$\begin{gathered} \varphi_j \in L_q(G),\ \text{supp}\, \varphi_j \subset G_j \subset G,\ 1/p + 1/q = 1, \varphi_j \in W_q^{\varepsilon_0}(G_j),\ \varepsilon_0 > 0, \\ a_\alpha \in C(\overline{Q}),\ a_\alpha \in L_\infty(0, T, C^{\varepsilon_0}(\overline{G_j})),\ |\alpha| = 2m,\ j = 1, \ldots, r, \\ a_\alpha \in L_p(Q)\ (|\alpha| < 2m),\ b_{j\beta} \in C^{2m - m_j, 1 - \frac{m_j}{2m}}(\overline{S}),\ |\beta| \leq m_j,\ j = 1, \ldots, m. \end{gathered} \tag{63}$$

The matrix B of dimension $r \times r$ has the rows

$$\int\limits_G (b_1(0,x), \varphi_k)\, dx, ..., \int\limits_G (b_r(0,x), \varphi_k)\, dx,\ k = 1, \ldots, r.$$

The claim of the previous theorem can be reformulated as follows.

Theorem 8. *Assume that the conditions* (36), (55), (56), (60), (63) *hold and the problem* (46) *satisfies the condition* (PL) *with the above-defined operator* A_0. *Then there exists a unique solution* $(u, q_1, .., q_{r_0})$ *to the problem* (30), (33), (62) *such that*

$$u \in W_p^{2m,1}(Q),\ q_i(t) \in L_p(0,T),\ i = 1, 2, ..., r.$$

A solution satisfies the estimate

$$\|u\|_{W_p^{2m,1}(Q)} + \sum_{i=1}^{r} \|q_i(t)\|_{C([0,T])} \leq$$

$$\leq c \left(\|f\|_{L_p} + \sum_{j=1}^{m} \|g_j\|_{W_p^{2mk_j,k_j}(S)} + \|u_0\|_{W_p^{2m-2m/p}(G)} + \sum_{j=1}^{s} \|\psi_i\|_{W_p^1(0,T)} \right).$$

Remark 4. The results of the above Theorems 1–8 remain valid in the case of unbounded domains G for which the solvability theorems of the direct problems are valid (the conditions on the coefficients slightly differ from the above-presented, see those in [83, Theorem 9.1], the Theorem 5.7 for $G = \mathbb{R}^n$, Theorem 7.11 for $G = \mathbb{R}^n_+$ in [82]. Note also that the results in [5–7] employ more general condition rather than the condition (A).

Remark 5. First, we note that the conditions on the lower order coefficients in Theorems 4, 5 can be weakened. It suffices to require that $a_{i\alpha} \in L_p(Q)$ or $a_\alpha \in L_p(Q)$ rather than $a_{i\alpha} \in L_\infty(Q)$ or $a_\alpha \in L_\infty(Q)$. Second, we can note that stability estimates for solutions similar to those in Theorems 1, 2 are also valid in all remaining theorems.

Acknowledgment. *The author was supported by the Russian foundation for basic research* (*Grant 18-01-00620a*).

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Received August 19, 2020

УДК 517.956 **DOI: 10.14529/mmp210101**

ОБ ЭВОЛЮЦИОННЫХ ОБРАТНЫХ ЗАДАЧАХ ДЛЯ МАТЕМАТИЧЕСКИХ МОДЕЛЕЙ ТЕПЛОМАССОПЕРЕНОСА

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Представлены результаты о корректности обратных задач для математических моделей тепломассопереноса. Неизвестными являются правая часть в уравнении (функция источников) и коэффициенты уравнения. Условия переопределения – значения решения на некоторых многообразиях или в отдельных точках. Рассматриваются два класса математических моделей. Первая включает систему уравнений Навье – Стокса, дополненную параболическим уравнением для температуры и параболической системой для концентраций примесей. Правая часть неизвестна и характеризует объемную плотность источников в жидкости. Неизвестные функции зависят от времени и части пространственных переменных и входят в правую часть уравнения. Второй класс систем – параболическая система уравнений уравнений для концентраций переносимых веществ, где неизвестные входят как в правую часть так и саму систему в качестве коэффициентов. Показана корректность этих задач, в частности полученные теоремы существования, единственности и оценки устойчивости для решений. Далее, мы опишем некоторые алгоритмы решения обратных задач о восстановлении точечных источников по точечным данным переопределения, основанные на асимптотике решений функций Грина соответствующих эллиптических задач.

Ключевые слова: обратная задача; тепломассообмен; фильтрация; диффузия; корректность.

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Поступила в редакцию 19 августа 2020 г.