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APPLICATION OF THE SMOOTH APPROXIMATION OF THE PROBABILITY FUNCTION IN SOME APPLIED STOCHASTIC PROGRAMMING PROBLEMS

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This paper is devoted to the application of the smooth approximation of the probability function in the solution of three different stochastic optimization problems: minimization of an airstrip area under the constrained probability of successful landing, minimization of the cost of water supply system with random performance and with predefined water consumption, and determination of the set of wind speed vectors which guarantees the safe landing of an aircraft in future with the given probability. The first two problems are mathematical programming problems with probability constraint, and the third one is a problem of constructing the isoquant surface of the probability function. Smooth approximation of the probability function allows to use the gradient projection method in the constrained optimization problem and to define the isoquant surface as the solution to a partial differential equation. We provide an example for each of the considered problems and compare the results with known results previously obtained using the confidence method.

Keywords: stochastic programming; probability function; sigmoid function; gradient projection method.

Dedicated to the 70-th anniversary of Professor A.I. Kibzun

1. Introduction

1.1. Overview of Stochastic Programming Solution Methods

The design of control systems in real-world problems requires considering the influence of the system environment, which often cannot be predicted at the design stage. Such design problems can be represented as mathematical programming problems, where the optimization vector represents some controlled parameters. One way to reflect the undetermined effects of the problem is to include random parameters in the model. This approach is not universal, since sometimes it is better to adapt the system to the worst possible influence. But it appears very effective in many engineering, financial, and social problems to include such undetermined effects as random parameters. The addition of the randomness in the model turns the objective and constraints in the optimization problem into random functions. So we need a special way to choose the optimal control, as we cannot compare objective values at different controls directly. One of the most reliable and sophisticated ways is to consider the optimization problems in which the objective or constraints are in the form of a probability function or a quantile function. This leads to stochastic optimization problems with probability and quantile functions [1].

Stochastic optimization problems are highly complex. A unified analytical method for the solution of these problems does not exist. Particular problems often require the development of a new solution technique or some kind of modification of the existing methods. The most universal method is the stochastic quasi-gradient method [1, 2], which is based on the estimation of the upper-bound value of function gradient and has a low rate of convergence. An approach involving sample average estimates was used for reducing a two-stage optimization problem [3] and can be used for other problems, but this method heavily depends on the type of the criterion and constraints functions.

One of the most powerful solution techniques is the confidence method. The method is based on a generalized minimax approach and its modifications [1, 4, 5]. The inner maximization of a minimax problem requires the selection of the optimal confidence set, which is a continual problem. This approach is hard to implement and usually leads to a suboptimal solution, as the optimization usually takes place only on the parameterized subset of confidence sets. One of the ways to enhance the confidence method is to consider sample probability space as if the random parameter distributions are discrete. This leads to the mixed integer programming problem [6, 7], thus eliminating the stochastic aspect of the problem.

One more type of solution method includes equivalent transformation methods [1], which aim is to reduce the problem to a more simple problem of another type. For example, some stochastic optimization problems with probability constraints can be reduced to the equivalent quantile optimization problem [1]. And in some cases, the objective in the form of probability or quantile function can be expressed as a usual function via probability density function or cumulative distribution function.

Another type of solution method is based on the calculation of the probability function gradient. The direct calculation of the gradient is a very complex problem since direct formulas imply the integration over the surface [8, 9] or the application of Lebesgue transformation [10]. The direct calculation can be implemented through volume integration [11], but only in limited cases. Other approaches to derive the gradient value approximations are described in [12–14], but the application of these approaches is limited by the type of distribution or other stochastic mechanisms.

1.2. Considered Problems and Solution Approach

In the present paper, we consider three problem statements, which were studied earlier [1], and provide a new solution technique based on our previous work [15].

The first task is to minimize the area of an airstrip, under the constraint that the probability of a successful landing of the aircraft must exceed a certain specified level. In the problem statement the random parameters describe the bias of an aircraft from the desired landing point due to the random speed and direction of the wind at the landing location.

The second problem is also related to the problem of the aircraft landing. Consider the aircraft that fly from airport A to airport B. We can measure the wind speed and direction at airport B and decide whether the landing is safe or not, but only at this moment. Usually, the decision about takeoff is based on the measured wind parameters at airport B just before the takeoff; if these parameters are not safe for landing, takeoff can be delayed, which can lead to financial losses. But even if wind parameters are safe for landing at the moment of takeoff, the speed and direction of the wind at airport B can changed during the flight, and new conditions could not be safe for landing at the moment of aircraft arrival at airport B. In this case, the aircraft is directed to the other airport, which also leads to financial losses. So the problem is to determine the set of possible wind speed vectors measured at the moment of takeoff from airport A, which guarantees that the probability of the safe landing exceeds a certain level.

The third problem statement describes the water supply system in a desert region. The system includes a water desalination plant, solar cells that provide energy for this plant, and a reservoir for freshwater storage. The price of such a system depends on the area of solar cells and the volume of the reservoir. The solar activity is considered to be random, which means that every month the amount of freshwater obtained from the plant is different. Also, it is possible to supply water from an external source in case of a freshwater shortage, but the price of such a supply is considered to be relatively high. So, the problem is to create the cheapest system that meets the requirements in freshwater with a high enough probability.

All these problems were solved using the confidence method [1], but this method, as we stated above, is rather hard to implement and leads to a suboptimal solution. We propose a new solution approach using the approximation of the probability function gradient as a basic tool of optimization. The idea of such approximation is to replace the Heaviside function inside the probability function with its smooth approximation - a sigmoid function [15]. Using this replacement, we obtain the approximation expressions for the probability function along with its gradient in a form of a volume integral, which is relatively simple to calculate. Also, this allows us to use methods of optimization involving the function gradient.

The paper is organized as follows. In Section 2 we provide all necessary formulas, relations, and statements. Section 3 presents the solution to the airstrip area optimization problem. Section 4 presents the problem of determining the set of admissible wind speed vectors. Section 5 describes the water supply problem. In Section 6 we give an overview of the paper and discuss the possible direction of the future work.

2. Related Results

First, we provide all necessary formulas and statements for the approximation of the probability function. Consider a complete probability space $(\mathbf{\Omega}, \mathbf{F}, \mathbf{P})$, an absolutely continuous variable X with the probability density function $f(x)$ on that space, and a smooth strictly piecewise monotonic function $g(u, x)$ depending on a control vector $u \in U,\ U \subset \mathbb{R}^m$. The probability that random value $g(u, X)$ does not exceed a specified level φ can be defined as

$$\mathbf{P}\left\{g(u,X) \leq \varphi\right\} = \int\limits_{-\infty}^{+\infty} I\left\{g(u,x) \leq \varphi\right\} f(x)dx = \int\limits_{-\infty}^{+\infty} \Theta(\varphi - g(u,x))f(x)dx, \qquad (1)$$

where $I(\cdot)$ is the indicator function and $\Theta(\cdot)$ is the Heaviside function. One of the problems in the probability function differentiation is that Heaviside function is discontinuous at zero. So, according to our previous work [15], we replace the Heaviside function with its smooth analogue - the sigmoid function - to get a differentiable approximation of the probability function. The sigmoid function is defined as follows

$$S_\theta(t) = \frac{1}{1 + e^{-\theta t}},$$

where the parameter θ corresponds to the steepness of sigmoid function. We showed that the probability function approximation (hereinafter referred to as $P_\theta(u)$) converges to the original probability function as the parameter θ approaches infinity, i.e.

$$\lim_{\theta\to+\infty} P_\theta(u) = \lim_{\theta\to+\infty} \int\limits_{-\infty}^{+\infty} S_\theta(\varphi - g(u,x))f(x)dx = \mathbf{P}\left\{g(u,X) \le \varphi\right\}. \qquad (2)$$

Also, we provide the approximation expressions for the probability function gradient and proof similar convergence statements for partial derivatives with respect to the specified loss level φ and with respect to the components of the control vector u_i, $i = \overline{1,m}$:

$$\frac{\partial}{\partial\varphi} P_\theta(u) = \int\limits_{-\infty}^{+\infty} \theta\left[1 - S_\theta(\varphi - g(u,x))\right] S_\theta(\varphi - g(u,x))f(x)dx, \qquad (3)$$

$$\lim_{\theta\to+\infty} \frac{\partial}{\partial\varphi} P_\theta(u) = \frac{\partial}{\partial\varphi}\mathbf{P}\left\{g(u,X) \le \varphi\right\}, \qquad (4)$$

$$\frac{\partial}{\partial u_i} P_\theta(u) = \int\limits_{-\infty}^{+\infty} \theta\left[S_\theta(\varphi - g(u,x)) - 1\right] S_\theta(\varphi - g(u,x))\frac{\partial g(u,x)}{\partial u_i} f(x)dx, \qquad (5)$$

$$\lim_{\theta\to+\infty} \frac{\partial}{\partial u_i} P_\theta(u) = \frac{\partial}{\partial u_i}\mathbf{P}\left\{g(u,X) \le \varphi\right\}. \qquad (6)$$

The same results are valid if the variable X is a random vector with absolutely continuous distribution and the probability density function $f_X(x) : \mathbb{R}^n \to \mathbb{R}^1$. In this case, the probability function approximation is defined as

$$P_\theta(u) = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} S_\theta\left(\varphi - g(u,x)\right) f_X(x)dx_n \ldots dx_2 dx_1. \qquad (7)$$

Using the Lebesgue dominated convergence theorem for conditional probability functions and joint probability density functions, we show that the same statements as (2)-(6) are valid in multidimensional case.

The same approach can be used if we need to handle the probability of several inequalities. For example, if we have two functions $g_1(u,X)$ and $g_2(u,X)$ and the desired probability is $\mathbf{P}\left(g_1(u,X) \ge 0, g_2(u,X) \ge 0\right)$, then it can be approximated in the same way

$$\lim_{\theta\to\infty} \int\limits_{-\infty}^{\infty} S_\theta(g_1(u,x))S_\theta(g_2(u,x))f(x)dx = \mathbf{P}\left(g_1(u,X) \ge 0, g_2(u,X) \ge 0\right). \qquad (8)$$

We need to ensure that the gradient of an objective function exists because we use the gradient projection method. If the objective function is the minimum or maximum of several other functions, we can not use the approximations because the derivative of the maximum or minimum function does not exist at the point of the intersection of nested functions. To solve this issue, we use smooth maximum or minimum transformation as the replacement of the usual maximum and minimum functions. The smooth minimum for the set of functions $y_1(x), y_2(x), ..., y_k(x)$ is defined as

$$SMin_\gamma(y_1(x), ..., y_k(x)) = \frac{\sum\limits_{i=1}^{k} y_i(x) e^{\gamma y_i(x)}}{\sum\limits_{i=1}^{k} e^{\gamma y_i(x)}}, \tag{9}$$

where $\gamma < 0$ is a large modulo number corresponding with the steepness of approximated function. The formula (9) remains the same for the case of smooth maximum, except the parameter γ must be a large positive number.

It is easy to show that the smooth minimum value converges to the value of the original minimum function:

$$\lim_{\gamma \to -\infty} SMin_\gamma(y_1(x), ..., y_k(x)) = \min(y_1(x), ..., y_k(x)). \tag{10}$$

The derivative of the smooth minimum function can be calculated directly:

$$\frac{d}{dx} SMin_\gamma(y_1(x), ..., y_k(x)) = \frac{\sum\limits_{i=1}^{k} y_i'(x) e^{\gamma y_i(x)}}{\sum\limits_{i=1}^{k} e^{\gamma y_i(x)}} + T(\gamma, y_1(x), ...y_k(x)), \tag{11}$$

and we can show that

$$\lim_{\gamma \to -\infty} T(\gamma, y_1(x), ...y_k(x)) = 0. \tag{12}$$

Furthermore, we can show that the derivatives of the smooth minimum converge to the derivatives of the minimum at the points where the derivative of the minimum exists, e.g.

$$\lim_{\gamma \to -\infty} \frac{d}{dx} SMin_\gamma(y_1(x), ..., y_k(x)) = \frac{d}{dx} \min(y_1(x), ..., y_k(x)) \tag{13}$$

at any point x where the right-hand side derivative is defined.

3. Airstrip Area Optimization

Consider the problem of a successful aircraft landing. We want to build the airstrip with minimal possible area to minimize the costs and we must meet the safety requirements. The airstrip must be long and wide enough that the aircraft that lands at some point of an airstrip stop without violating the boundaries. The minimal possible length of an airstrip must be equal to the free path of an aircraft from the touchdown point till the full stop. The minimal possible width of an airstrip must be equal to the distance between the endpoints of the chassis.

In a real situation, the aircraft is affected by the wind, so we cannot guarantee that the aircraft lands at the desired point. We must ensure that the airstrip margins are big enough to compensate for the bias with a high probability. Under the assumption that the wind speed is random and it has some distribution with infinite support, we cannot guarantee the successful landing with probability 1, because it requires the infinite airstrip.

This is the motivation to select the desired probability at some high enough level, e.g. 0,9, 0,999, or even 0,99999 (this level is suitable for civil aviation).

Let us denote by l_0 the distance from the aircraft landing point to the point of complete stop, measured without external factors. The actual touchdown point of the aircraft will differ from the desired one, so let l_1 and l_2 denote the runway front and back length margins. Let us also denote by z_1 the half of the airstrip width. For simplicity, assume that the distance between the endpoints of the chassis is equal to zero. Then the area of an airstrip, which is the objective function of the problem, can be defined as follows

$$S(z_1, l_1, l_2) = 2z_1 \left(l_0 + l_1 + l_2\right). \tag{14}$$

The stochastic model in the problem statement describes the random shift from the desired landing point due to the random speed and direction of the wind at the landing location. Let W_x and W_z denote the longitudinal and transverse components of the wind vector at the landing location. Assume that W_x and W_z are independent random variables with Gaussian distribution with known parameters:

$$W_x \sim \mathcal{N}(m_x, \sigma_x^2), \quad W_z \sim \mathcal{N}(m_z, \sigma_z^2). \tag{15}$$

Let X and Z denote the longitudinal and transverse shift from the landing point. These shifts are related to wind components as follows:

$$X = a_{11}W_x + a_{12}|W_z|, \quad Z = a_{22}W_z, \tag{16}$$

where a_{11}, a_{12}, a_{22} are known coefficients. The landing is considered successful if

$$-l_1 \leq X \leq l_2, \quad |Z| \leq z_1. \tag{17}$$

These inequalities must hold with a probability not less than the desired level of reliability, it is the safety requirement. Hereinafter the level of reliability is denoted by α.

It is worth noting that we will optimize the logarithm of an objective function (14) to reduce the scale of parameters. So, the problem is defined as

$$\ln z_1 + \ln\left(l_0 + l_1 + l_2\right) \to \min \tag{18}$$

with probability constraint

$$\mathbf{P}\left\{-l_1 \leq X \leq l_2, \quad |Z| \leq z_1\right\} \geq \alpha. \tag{19}$$

To solve the optimization problem with probability constraint we use the gradient projection method with decaying steps. This algorithm was implemented in the Python programming language.

Example 1. The problem parameters were set according to [1]:

$$a_{11} = a_{12} = -20\,\text{sec},\ a_{22} = 3\,\text{sec},\ l_0 = 1500\,\text{m},\ \sigma_x = \sigma_z = 5\,\frac{\text{m}}{\text{sec}},$$
$$m_x = m_z = 0,\ \alpha = 0,99.$$

The calculation required 7,59 seconds; the results along with the solutions obtained by other methods [1] are presented in the Table.

As we can see from the Table, the new approach gives a smaller optimal area comparing to the results of the confidence method. But this area is quite bigger than the one obtained with the quasi-gradient method. The difference between these two solutions appears probably due to the non-optimal weight decay ratio. This might be the subject of further analysis.

Table

Results of calculation and other solutions

Solution method	Optimal area S, km^2	l_1, m	l_2, m	z_1, m
Proposed algorithm	0,185	583,1	283,1	39,2
Confidence method	0,202	430,2	290,1	45,54
Quasi-gradient algorithm	0,170	400,3	213,2	40,23

4. Wind Speed Forecasting

Consider the situation when the aircraft is ready for take-off at airport A with a destination at airport B. Before the take-off, we can measure the wind speed and direction at the destination airport B and decide whether the landing will be safe. But during the flight, both the speed and direction of the wind can changed. If the conditions become non-safe, the aircraft is redirected to the backup airport, which leads to financial losses. The problem is to determine the set of possible wind speed vectors, which guarantees that the landing will be safe with the given probability.

Let v_0 and β_0 denote the initial wind speed and direction (that is angle in radians) at the destination airport. The longitudinal component W_x^0 and transverse component W_z^0 of the wind speed vector at the time of departure are

$$W_x^0 = v_0 \cos(\beta_0), \quad W_z^0 = v_0 \sin(\beta_0). \tag{20}$$

Let ξ and η be the variables controlling the wind speed and direction shifts. We assume that these variables are independent and have Gaussian distribution:

$$\xi \sim \mathcal{N}(0, \sigma_\xi^2), \quad \eta \sim \mathcal{N}(0, \sigma_\eta^2). \tag{21}$$

The longitudinal component W_x^t and transverse component W_z^t of the wind speed vector at the time of departure and at the time of arrival are related as

$$W_x^t = (v_0 + \xi)\cos(\beta_0 + \eta), \tag{22}$$
$$W_z^t = (v_0 + \xi)\sin(\beta_0 + \eta). \tag{23}$$

Let $w_x^{\max}$ and $w_x^{\min}$ denote the maximum and minimum permissible speeds for the longitudinal wind (tailwind and headwind), and $w_z^{\max}$ denote the maximum permissible cross-runway wind speed. Upon approaching the destination airport, the plane will receive a landing permit only if the following condition is met:

$$|W_z^t| \le w_z^{\max}, w_x^{\min} \le W_x^t \le w_x^{\max}. \tag{24}$$

Let us denote this probability as the function of v_0 and β_0:

$$P(v_0, \beta_0) = \mathbf{P}\left(|W_z^t| \le w_z^{\max}, w_x^{\min} \le W_x^t \le w_x^{\max}\right). \tag{25}$$

We need to find all points (v_0, β_0), at which the following inequality is fulfilled:

$$P(v_0, \beta_0) \ge \alpha, \tag{26}$$

where α is the desired level of reliability.

The set of admissible wind speed vectors should be a closed convex set, containing point zero, so that any affine combination of these vectors should give an admissible wind speed vector. We can find the isoquant of the probability function $P(v_0, \beta_0)$ to find the boundary of this set. First, we need to find the starting point with coordinates v_0, β_0. For example, we let $\beta_0 = 0$ and numerically solve the equation

$$P(v_0, 0) = \alpha. \tag{27}$$

Using this starting point, we rotate the wind speed vector with a given step $\Delta\beta$ and find the wind speed change Δv for which the probability remains unchanged, so the following equality must hold:

$$\frac{\partial}{\partial \beta_0} P(v_0, \beta_0)\Delta\beta + \frac{\partial}{\partial v_0} P(v_0, \beta_0)\Delta v = 0. \tag{28}$$

The border of the set of admissible wind speed vectors is the solution of the equation

$$\frac{dv}{d\beta} = -\frac{\frac{\partial}{\partial \beta} P(v, \beta)}{\frac{\partial}{\partial v} P(v, \beta)}, \tag{29}$$

with initial condition

$$v(\beta_0) = v_0. \tag{30}$$

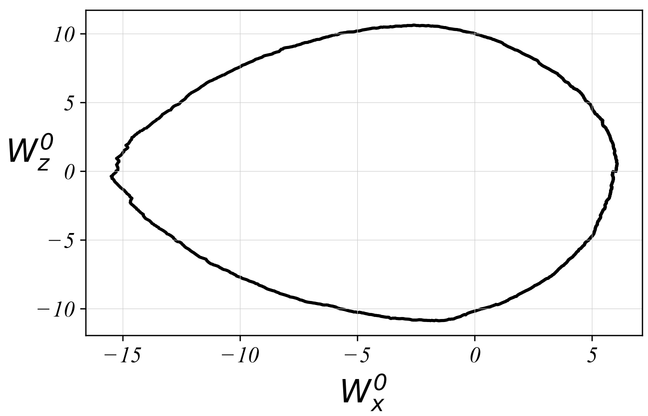
Example 2. The initial data was set according to [1]:

$$\sigma_\xi = 1,9\,\frac{\text{m}}{\text{sec}},\ \sigma_\eta = 27^\circ,$$

which corresponds with the 3-hour wind change in the area of the Moscow Region airport,

$$\alpha = 0,99,\ w_z^{\max} = 15\,\frac{\text{m}}{\text{sec}},\ w_x^{\min} = -25\,\frac{\text{m}}{\text{sec}},\ w_x^{\max} = 10\,\frac{\text{m}}{\text{sec}}.$$

The initial point is $v_0 = 6,101$, $\beta_0 = 0$. The calculation required 36,8 seconds. The result set is presented at the Figure.



The border of the set of the admissible wind speeds

Comparing to the results from [1], we obtained a convex set. The set in [1] was constructed using the confidence method with the union of two specific confidence sets: an ellipse and a rectangle.

The same problem was considered as an example in [16], which was dedicated to the problem of the construction of the confidence absorbing set for the analysis of static stochastic systems. The results of these two approaches are similar, but the approach that is proposed in the present article seems a bit easier.

5. Design of Water Supply System

Consider the water supply system in a desert region. We assume that the spring with the salt water is nearby, so the main part of the system is the water desalination plant. This plant is powered by energy generated from the solar cells, and obtained freshwater is stored in the reservoir. The price of such a system depends on the area of solar cells and the volume of the reservoir. If the freshwater demand is not satisfied, we assume that it is possible to bring freshwater from a distant external source, but the price of such supply is considered relatively high. We also assume that in each month the water demand is known beforehand, but the solar activity is random. This fact affects the performance of solar cells thus affecting the freshwater generation. The problem is to create the cheapest system that meets the requirements in freshwater demand with a high enough probability.

Let t be the number of months. Let d_j denote the freshwater demand in j-th month, and u_j denote the amount of freshwater brought from an external source in j-th month. The area of the solar cell is denoted by s and the reservoir volume is denoted by v. Let a_0 denote the cost of 1 cubic meter of freshwater from an external source, a_1 denote the cost of 1 square meter of solar cell and a_2 denote the cost of 1 cubic meter of the reservoir.

The performance of the solar cell in the j-th month we denote by X_j. We assume that the variables $\{X_j\}_{j=1}^t$ are independent and each variable X_j has Gaussian distribution with a known expectation and variance:

$$X_j \sim \mathcal{N}(m_j, \sigma_j^2) \quad \forall j \in \overline{1,t}. \tag{31}$$

The overall cost of the system is defined as

$$\Phi_0(u) \triangleq a_1 s + a_2 v + a_0 \sum_{j=1}^{t} u_j, \tag{32}$$

where $u \triangleq (s, v, u_1, \ldots, u_t)^T$. The objective is to minimize the cost of the system.

Now we define the constraints of the problem. The amount of freshwater left after the month j is defined as

$$Z_j \triangleq \min\{Z_{j-1}, v\} + sX_j + u_j - d_j, \quad Z_0 = 0, \quad j = \overline{1,t}, \tag{33}$$

where $\min\{Z_{j-1}, v\}$ is the amount of freshwater left after previous month. The freshwater surplus is stored in the reservoir and can be used to cover the shortage in future months. The amount of freshwater left after the first month is

$$Z_1 = X_1 s + u_1 - d_1. \tag{34}$$

The system must satisfy the water demand with the given probability:

$$\mathbf{P}\left(Z_j(u, X) \geq 0,\, j = \overline{1,t}\right) \geq \alpha. \tag{35}$$

This probability can be rewritten as

$$\mathbf{P}\left(Z_j(u,X) \geq 0,\ j=\overline{1,t}\right) = \mathbf{P}\left(\min_{j=\overline{1,t}} Z_j(u,X) \geq 0\right). \tag{36}$$

To obtain the differentiable approximation of this constraint, we replace the outer and inner minimums with the smooth minimum (9) and replace the whole probability function with its smooth approximation (2).

So the final problem is to minimize the price of the system

$$a_1 s + a_2 v + a_0 \sum_{j=1}^{t} u_j \to \min_u \tag{37}$$

under probability constraint

$$\int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} S_\theta\left(SMin\left[Z_1(u,x),\ldots,Z_t(u,x)\right]\right) f_X(x) dx_t \ldots dx_1 \geq \alpha. \tag{38}$$

Example 3. To give an example we use the parameters from [1]. We set $\alpha = 0,99$, $t = 6$, $a_0 = 250\,\$/\mathrm{m}^3$, $a_1 = 37,5\,\$/\mathrm{m}^2$, $a_2 = 100\,\$/\mathrm{m}^3$, $d_1 = 29,6\,\mathrm{m}^3$, $d_2 = 23,9\,\mathrm{m}^3$, $d_3 = 36,2\,\mathrm{m}^3$, $d_4 = 82,1\,\mathrm{m}^3$, $d_5 = 96,5\,\mathrm{m}^3$, $d_6 = 173,4\,\mathrm{m}^3$, $m_1 = 0,0084$, $m_2 = 0,0083$, $m_3 = 0,0185$, $m_4 = 0,0063$, $m_5 = 0,123$, $m_6 = 0,137$, $\sigma_1^2 = 5,8 \cdot 10^{-4}$, $\sigma_2^2 = 5,5 \cdot 10^{-4}$, $\sigma_3^2 = 12,3 \cdot 10^{-4}$, $\sigma_4^2 = 42,1 \cdot 10^{-4}$, $\sigma_5^2 = 81,8 \cdot 10^{-4}$, $\sigma_6^2 = 91,6 \cdot 10^{-4}$. All probabilities and expectations were calculated using the Monte-Carlo method with a sample size of 10000.

The calculation required 292,69 seconds. The optimal cost of the system is $100613,5\,\$$, the area s of the solar cell is $2488,1\,\mathrm{m}^2$, the volume v of the reservoir is $31,4\,\mathrm{m}^3$, $u_1 = 15,3$, $u_2 = 1,3$, $u_3 = u_4 = u_5 = u_6 = 0$. So, the supply of freshwater from external source is required only for the first two months.

We do not compare directly this result to the one obtained in [1], since it is not correct. The area of a solar cell obtained in [1] is $1000\,m^2$, which is not enough to satisfy the demand even in the case of expected solar activity in each month.

Conclusion

In the present paper, we consider three applied stochastic programming problems: minimization of an airstrip area under the constrained probability of successful landing, minimization of the cost of water supply system with random performance and given water consumption, and determination of the set of wind speed vectors, measured at the moment of departure, which guarantees the safe landing with a given probability. We show that the smooth approximation of the probability function is a powerful tool for solving different problem types. It allows us to use the gradient projection method in the constrained optimization problem, and to define the isoquant surface as the solution of a partial differential equation. The same approach can be used in problems with an objective in form of the probability function, and problems with mixed probability and deterministic constraints. Formulas are presented as the volume integrals, so they can be efficiently calculated. Further work may include a generalization of this approach for other types of problems and improvement of the established solution algorithms.

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О РЕШЕНИИ НЕКОТОРЫХ ПРИКЛАДНЫХ ЗАДАЧ СТОХАСТИЧЕСКОГО ПРОГРАММИРОВАНИЯ С ПОМОЩЬЮ ГЛАДКОЙ АППРОКСИМАЦИИ ФУНКЦИИ ВЕРОЯТНОСТИ

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В статье описано применение гладкой аппроксимации функции вероятности в трех прикладных задачах стохастического программирования: задаче минимизации площади взлетно-посадочной полосы при ограничении на вероятность успешной посадки, задаче минимизации стоимости системы обеспечения пресной водой в условиях случайной производительности и заданного потребления воды, а также задаче определения множества допустимых скоростей ветра, при которых с заданной вероятностью можно обеспечить безопасную посадку самолета по прошествии времени полета. Первые две задачи являются задачами оптимизации с вероятностным ограничением, третья задача сводится к задаче определения поверхности уровня функции вероятности. Гладкая аппроксимация функции вероятности позволяет использовать метод проекции градиента в задачах условной оптимизации, а также позволяет свести задачу построения линии уровня функции вероятности к решению уравнения в частных производных. Все задачи сопровождаются расчетными примерами. Полученные результаты сравниваются с решениями, полученными ранее с помощью доверительного метода.

Ключевые слова: стохастическое программирование; функция вероятности; сигмоидальная функция; метод проекции градиента.

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