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INVARIANT MANIFOLDS OF THE HOFF MODEL IN "NOISE" SPACES

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The work is devoted to the study the stochastic analogue of the Hoff equation, which is a model of the deviation of an I-beam from the equilibrium position. The stability of the model is shown for some values of the parameters of this model. In the study, the model is considered as a stochastic semilinear Sobolev type equation. The obtained results are transferred to the Hoff equation, considered in specially constructed "noise" spaces. It is proved that, in the vicinity of the zero point, there exist finite-dimensional unstable and infinite-dimensional stable invariant manifolds of the Hoff equation with positive values of parameters characterizing the properties of the beam material and the load on the beam.

Keywords: the Nelson–Gliklikh derivative; stochastic Sobolev type equations; invariant manifolds.

Introduction

The Hoff model

$$(\lambda + \Delta)\dot{u} = \alpha u + \beta u^3, \tag{1}$$

$$u(x,0) = u_0,\ u \in \Sigma,\ u(t,0) = 0, (x,t) \in \partial\Sigma \times \mathbb{R} \tag{2}$$

is a model of buckling of an I-beam from the equilibrium position. Here $\Sigma \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Sigma$, the parameter $\lambda \in \mathbb{R}_+$ is the parameter responsible for the load applied to the beam and $\alpha,\ \beta \in \mathbb{R}$ are the parameters responsible for the material from which the beam is made. The paper [1] considers the set of valid initial data of problem (1), (2) understood as a phase space. Here equation (1) was reduced to the semilinear Sobolev type equation

$$L\dot{u} = Mu + N(u), \tag{3}$$

where $L,\ M,\ N : \mathfrak{U} \to \mathfrak{F}$ are the operators and $\mathfrak{U},\ \mathfrak{F}$ are Banach spaces selected in a special way. In [2], the proof of smoothness and simplicity of the phase space of the equation for positive values of the parameters α and β is considered. The stability of solutions to equation (1) in a neighborhood of the zero point is described in [3], which shows the existence of stable and unstable invariant manifolds.

The purpose of this paper is to study the stability of the stochastic analogue of equation (1). We consider the Hoff equation as a special case of the stochastic semilinear Sobolev type equation

$$L \overset{o}{\eta} = M\eta + N(\eta). \tag{4}$$

Here, $\overset{o}{\eta}$ denotes the Nelson–Gliklikh derivative [4]. Currently, a large number of papers are devoted to the problem on the solvability of a linear ($N \equiv \mathbb{O}$) equation of the form

(4). Let us note only some of them. The paper [5] considers the existence of solutions to the Cauchy problem

$$\lim_{t\to 0+}(\eta(t)-\eta_0)=0, \tag{5}$$

the Showalter–Sidorov problem

$$P(\eta(0)-\eta_0)=0 \tag{6}$$

for linear equation (4) ($N\equiv\mathbb{O}$) in the case of the (L,p)-bounded operator M, $p\in\{0\}\bigcup\mathbb{N}$. Investigation of problems (5), (6) for equation (4) if $N\equiv\mathbb{O}$ in the case of the relative sectorial operator M is presented in [6], and in the case of the relative radial operator M is considered in [7].

The paper [8] considers the study of the nonlinear stochastic Sobolev type equation

$$L\overset{o}{\eta}=N(\eta). \tag{7}$$

The paper establishes the conditions for the existence of solutions to equation (7). In our study, the question of the stability of a semilinear equation of the form (7) is solved. In the linear case ($N\equiv\mathbb{O}$), the existence of stable and unstable invariant spaces was shown in [9]. This work is a continuation of [8, 9] on the study of local stability of a semilinear stochastic equation.

The paper is organized as follows. Section 1 contains some concepts and statements on the theory of stability of Sobolev type equations. In Section 2, we describe differential forms with coefficients from a specially selected “noises” spaces obtained by Nelson–Gliklikh derivative. In this sections, we research the exponential dichotomy of linear Sobolev type equations and invariant manifolds of semilinear Sobolev type equations. In Section 3, we present an example for the stochastic analogue of the Hoff equation.

1. Invariant Manifolds of Sobolev Type Equations

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, $L,\ M\in\mathcal{L}(\mathfrak{U};\mathfrak{F})$ be operators. The set $\rho^L(M)=\{\mu\in\mathbb{C}:(\mu L-M)^{-1}\in\mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ is called the *L-resolvent set* and the set $\sigma^L(M)=\mathbb{C}\setminus\rho^L(M)$ is called the *L-spectrum of the operator M*. The operator M is called the (L,σ)-*bounded operator*, if $\sigma^L(M)$ is bounded.

Let M be a (L,σ)-bounded operator. Then there exist a splitting of the spaces $\mathfrak{U}^0$ $(\mathfrak{U}^1)=\ker P$ $(\mathrm{im}P)$, $\mathfrak{F}^0$ $(\mathfrak{F}^1)=\ker Q$ $(\mathrm{im}Q)$, the operators L_k $(M_k)\in\mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$ $(k=0,1)$, and the operators $M_0^{-1}\in\mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$ and $L_1^{-1}\in\mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$ (see, for example, [10]). Here

$$P=\frac{1}{2\pi i}\int_\gamma(\mu L-M)^{-1}Ld\mu\in\mathcal{L}(\mathfrak{U}),\quad Q=\frac{1}{2\pi i}\int_\gamma L(\mu L-M)^{-1}d\mu\in\mathcal{L}(\mathfrak{F})$$

are projectors, and the closed contour $\gamma\subset\mathbb{C}$ bounds a domain containing $\sigma^L(M)$. Consider the operators $H=L_0^{-1}M_0\in\mathcal{L}(\mathfrak{U}^0)$ and $S=L_1^{-1}M_1\in\mathcal{L}(\mathfrak{U}^1)$. If the operator M is (L,σ)-bounded operator and $H\equiv\mathbb{O}$, $p=0$ or $H^p\neq\mathbb{O}$, $H^{p+1}\equiv\mathbb{O}$, then the operator M is called (L,p)-*bounded* operator.

The vector function $u\in C^k((-\tau,\tau);\mathfrak{U})$, $k\in\mathbb{N}\cup\{\infty\}$ satisfying equation (3) for some $\tau\in\mathbb{R}_+$ is called a *solution* to this *equation.* The solution $u=u(t)$ to equation (3) is called a *solution to the Cauchy problem*

$$u(0)=u_0 \tag{8}$$

for equation (3), if equality (8) is satisfied for some $u_0\in\mathfrak{U}$.

Definition 1. *The set* $\mathfrak{P} \in \mathfrak{U}$ *is called the phase space of equation (3) if*

(i) any solution $u = u(t)$ *to equation (3) belongs to* $\mathfrak{P}$*, i.e.* $u(t) \in \mathfrak{P}$ *for every* $t \in (-\tau, \tau)$*;*

(ii) for any $u_0 \in \mathfrak{P}$*, there exists a unique solution* $u \in C^k((-\tau, \tau); \mathfrak{U})$*,* $k \in \mathbb{N} \cup \{\infty\}$ *to Cauchy problem (8) for equation (3).*

Cauchy problem (8) for equation (3) can be either unsolvable in general, or solvable, but not uniquely, even in the case when ∞ is a pole of the order $p \in \{0\} \cup \mathbb{N}$ of the L-resolvent of the operator M. Starting from the paper [1], in order to study the solvability of Cauchy problem (8) for equation (3), it is proposed to limit to *quasi-stationary trajectories*, i.e. such solutions to equation (3) for which $H\dot{u}^0(t) = \mathbb{O}$. These solutions belong to the set

$$\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(Mu + N(u)) = 0\}.$$

Note that if the operator $N = \mathbb{O}$, then the set $\mathfrak{M} = \mathfrak{U}^1$.

Let $u \in \mathfrak{M}$. The set $\mathfrak{M}$ is called the C^l-*manifold* at the point u, if there exist neighborhoods $\mathfrak{O} \subset \mathfrak{M}$ and $\mathfrak{O}^1 \subset \mathfrak{U}^1$ of the points $u \in \mathfrak{M}$ and $u^1 = Pu \in \mathfrak{U}^1$, respectively, and the C^l-diffeomorphism $D : \mathfrak{O}^1 \to \mathfrak{O}$ such that D^{-1} is a restriction of the projector P by $\mathfrak{M}$, $l \in \mathbb{N} \cup \{\infty\}$. The pair $(D, \mathfrak{O})$ is called a *map* of the set $\mathfrak{M}$. The set $\mathfrak{M}$ is called a *Banach* C^l*-manifold* if it is such at each of its points. A connected Banach C^l-manifold is called a *simple manifold*, if any of its atlases is equivalent to an atlas containing a single map.

Theorem 1. [1] *Let* M *be a* (L, p)*-bounded operator,* $p \in \{0\} \cup \mathbb{N}$*, the operator* $N \in C^k(\mathfrak{U}, \mathfrak{F})$*, and the set* $\mathfrak{M}$ *be a simple Banach* C^l*-manifold at the point* u_0*. Then, for some* $\tau \in \mathbb{R}_+$*, there exists a unique solution* $u \in C^m((-\tau, \tau); \mathfrak{M})$*,* $m = \min\{k, l\}$*, to equation (3) passing through the point* u_0*.*

Remark 1. If the operator $N = \mathbb{O}$, then the set $\mathfrak{M} = \mathfrak{U}^1$ and the phase space of the equation

$$L\dot{u} = Mu \tag{9}$$

is a subspace of $\mathfrak{U}^1$.

Definition 2. *If, for any solution* $u_0 \in \mathfrak{I} \subset \mathfrak{U}$ *to problem (9), (8) is* $u \in C^1(\mathbb{R}; \mathfrak{I})$*, then the space* $\mathfrak{I}$ *is called an invariant space of equation (9).*

Remark 2. For the existence of invariant spaces, it is sufficient to fulfill the condition

$$\left.\begin{array}{c} \sigma^L(M) = \sigma^L_1(M) \bigcup \sigma^L_0(M), \quad \sigma^L_1(M) \neq \oslash, \\ \sigma^L_1(M) \textit{ is a closed set.} \end{array}\right\} \tag{10}$$

Remark 3. Any invariant space $\mathfrak{I}$ of the equation (7) is a subspace of its phase space.

Definition 3. *If there exist constants* $N_{1(2)}$*,* $\nu_{1(2)} \in \mathbb{R}_+$ *and*

$$\begin{array}{c} \|u^1(t)\|_{\mathfrak{U}} \leq N_1 e^{-\nu_1(s-t)} \|u^1(s)\|_{\mathfrak{U}} \quad \textit{for} \quad s \geq t \quad (u^1 \in \mathfrak{I}^+) \\ (\|u^2(t)\|_{\mathfrak{U}} \leq N_2 e^{-\nu_2(t-s)} \|u^2(s)\|_{\mathfrak{U}} \quad \textit{for} \quad t \geq s \quad (u^2 \in \mathfrak{I}^-)), \end{array}$$

then invariant space $\mathfrak{I}^{+(-)} \subset \mathfrak{P}$ *is called a stable (unstable) invariant space of equation (9).*

Remark 4. (i) If $\mathfrak{I}^+ = \mathfrak{P}$ ($\mathfrak{I}^- = \mathfrak{P}$), then we talk about the stability (unstability) of the stationary solution to equation (9).

(ii) If $\mathfrak{P} = \mathfrak{I}^+ \oplus \mathfrak{I}^-$, then there exists an exponential dichotomy of solutions to equation (9).

Let the following condition be fulfilled:

$$\left.\begin{array}{c}\sigma^L(M) = \sigma^L_+(M) \bigcup \sigma^L_-(M) \ \textit{and} \\ \sigma^L_{+(-)}(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\mu > (<) 0\},\ \sigma^L_{+(-)}(M) \neq \oslash \end{array}\right\}. \tag{11}$$

Then we can construct the projectors

$$P_{l(r)} = \frac{1}{2\pi i} \int\limits_{\gamma_{l(r)}} R^L_\mu(M) d\mu,\ Q_{l(r)} = \frac{1}{2\pi i} \int\limits_{\gamma_{l(r)}} L^L_\mu(M) d\mu,$$

where the contour $\gamma_{l(r)}$ belongs to the left (right) half-plane and bounds the domain containing the part of the L-spectrum of the operator M which belongs to this half-plane.

Theorem 2. [4] *Let M be the (L,p)-bounded operator and condition (11) be fulfilled. Then there exist the stable $\mathfrak{I}^+ = \mathrm{im}P_l$ and unstable $\mathfrak{I}^- = \mathrm{im}P_r$ invariant spaces of equation (9).*

Definition 4. *The set*

$$\mathfrak{M}^{+(-)} = \{u_0 \in \mathfrak{U} : \|P_{l(r)}u_0\|_{\mathfrak{U}} \leq R_1,\ \|u(t,u_0)\|_{\mathfrak{U}} \leq R_2,\ t \in \mathbb{R}_{+(-)}\}$$

is such that

(i) $\mathfrak{M}^{+(-)}$ is diffeomorphic to a closed ball in $\mathfrak{I}^{+(-)}$;

(ii) $\mathfrak{M}^{+(-)}$ touches $\mathfrak{I}^{+(-)}$ at the zero point;

(iii) for any $u_0 \in \mathfrak{M}^{+(-)}$ and for $t \to +(-)\infty$, $\|u(t,u_0)\|_{\mathfrak{U}} \to 0$ is called a stable (unstable) invariant manifold of equation (3).

Here $u(t,u_0)$ is a quasi-stationary trajectory of equation (3) passing through the point $u_0 \in \mathfrak{M}$.

Theorem 3. [3] *Let M be the (L,p)-bounded operator, $p \in \{0\} \cup \mathbb{N}$, condition (11) be fulfilled, and the operator $N \in C^\infty(\mathfrak{U}, \mathfrak{F})$ be such that $N(0) = 0$, $N'_0 = \mathbb{O}$. Then for some R_j, $j = 1, 2$ there exist the stable and unstable invariant manifolds of equation (3). Moreover, if for some $u_0 \in \mathfrak{M}$, there exist $\|P_{l(r)}u_0\|_{\mathfrak{U}} \leq R_1$ and $\|u(t,u_0)\|_{\mathfrak{U}} \leq R_2$ for $t \to +(-)\infty$, then $u_0 \in \mathfrak{M}^{+(-)}$.*

2. Stable and Unstable Invariant Manifolds in "Noise" Spaces

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space and $\mathbf{L_2}$ be a set of random variables $\xi : \Omega \to \mathbb{R}$, whose mathematical expectation is zero ($\mathbf{E}\xi = 0$) and the variance ($\mathbf{D}$) is finite. In $\mathbf{L_2}$, we define the scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. Denote by $\mathbf{L_2^0} \subset \mathbf{L_2}$ the subspace of random variables measurable with respect to $\mathcal{A}_0$, where $\mathcal{A}_0$ is a σ-subalgebra of the σ-algebra $\mathcal{A}$. The orthoprojector $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$ is called a *conditional mathematical expectation* and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$.

The mapping $\eta : \mathbb{R} \times \Omega \to \mathbb{R}$ is called a *stochastic process*. If we fix $t \in \mathcal{J} \subset \mathbb{R}$, then the stochastic process $\eta = \eta(t, \cdot)$ is a random variable. If we fix $\omega \in \Omega$, then the stochastic process $\eta = \eta(\cdot, \omega)$ is called a *trajectory*. If almost certainly all the trajectories of the stochastic process η are continuous (i.e. for almost all $\omega \in \Omega$ the trajectories of $\eta(\cdot, \omega)$ are continuous), then η is called a *continuous* process. Denote by $\mathbf{CL_2}$ the set of continuous process. Fix $\eta \in \mathbf{CL_2}$ and $t \in \mathcal{J}$, and denote by $\mathcal{N}_t^\eta$ the σ-algebra generated by a random variable $\eta(t)$ and $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$.

Definition 5. [4] *Let $\eta \in \mathbf{CL_2}$. If there exists the limit*

$$\overset{o}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \boldsymbol{E}_t^\eta\left(\frac{\eta(t+\Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \boldsymbol{E}_t^\eta\left(\frac{\eta(t, \cdot) - \eta(t-\Delta t, \cdot)}{\Delta t}\right)\right),$$

then $\overset{o}{\eta}$ is called the Nelson–Gliklikh derivative of the stochastic process η at the point $t \in \mathcal{J}$.

Denote by $\mathbf{C}^l\mathbf{L}_2$, $l \in \mathbb{N}$ the space of stochastic processes whose trajectories are almost certainly differentiable by Nelson–Gliklikh on the interval $\mathcal{J}$ up to the order l inclusively. The spaces $\mathbf{C}^l\mathbf{L_2}$ are called the *spaces of differentiable "noises"*.

Let $\mathfrak{U}$ ($\mathfrak{F}$) be a real separable Hilbert space with a basis $\{\varphi_k\}$ ($\{\psi_k\}$) orthonormal with respect to the scalar product $< \cdot, \cdot >_\mathfrak{U}$ ($< \cdot, \cdot >_\mathfrak{F}$). Choose the sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < \infty$, and the sequence $\{\xi_k\} \subset \mathbf{L_2}$ ($\{\zeta_k\} \subset \mathbf{L_2}$) of uniformly bounded random variables. Next, we construct the *random* $\mathbf{K}$*-value*

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k \quad \left(\zeta = \sum_{k=1}^{\infty} \lambda_k \zeta_k \psi_k\right).$$

The completion of the linear shell with the random $\mathbf{K}$-values according to the norm

$$\|\xi\|^2_{\mathbf{U_K L_2}} = \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k \quad \left(\|\zeta\|^2_{\mathbf{F_K L_2}} = \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\zeta_k\right)$$

is a Hilbert space, which we denote by $\mathbf{U_K L_2}$ ($\mathbf{F_K L_2}$) and call the *space of random* $\mathbf{K}$*-values.*

The stochastic process $\eta : (\varepsilon, \tau) \to \mathbf{U_K L}_2$ is defined by the formula

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k \xi_k(t) \varphi_k, \tag{12}$$

where $\{\xi_k\}$ is some sequence from $\mathbf{CL}_2$ and $\mathcal{J} = (\varepsilon, \tau) \subset \mathbb{R}$, which is called a *stochastic continuous* $\mathbf{K}$*-process*, if the number on the right side converges uniformly on any compact set in $\mathcal{J}$ with the norm $\|\cdot\|_{\mathbf{U_K L_2}}$, and the trajectory of the process $\eta = \eta(t)$ is almost surely continuous. A continuous stochastic $\mathbf{K}$-process $\eta = \eta(t)$ is called a *process continuously differentiable by Nelson–Gliklikh on* $\mathcal{J}$, if the series

$$\overset{o}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k \overset{o}{\xi}_k(t) \varphi_k \tag{13}$$

converges on any compact in $\mathcal{J}$ according to the norm $\|\cdot\|_\mathbf{U}$ and the trajectories of the process $\overset{o}{\eta} = \overset{o}{\eta}(t)$ are almost certainly continuous. The symbol $\mathbf{C}(\mathcal{J}, \mathbf{U_K L}_2)$ denotes the space of continuous stochastic $\mathbf{K}$-processes and the symbol $\mathbf{C}^l(\mathcal{J}, \mathbf{U_K L}_2)$ denotes the space of the stochastic $\mathbf{K}$-processes continuously differentiable up to the order $l \in \mathbb{N}$. Examples of the vector space $\mathbf{C}^l\mathbf{L}_2$ and the stochastic $\mathbf{K}$-process continuously differentiable up to any order $l \in \mathbb{N}$ are given by the stochastic process describing Brownian motion in the Einstein–Smolukhovsky model

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin \frac{\pi}{2}(2k+1)t,$$

where $\xi_k \in \mathbf{L}_2$, $D\xi_k = [\frac{\pi}{2}(2k+1)]^{-2}$, $k \in \{0\} \cup \mathbb{N}$, $\overset{o}{\beta}(t) = \frac{\beta(t)}{2t}, t \in \mathbb{R}_+$, and the Wiener's $\mathbf{K}$-process

$$W_{\mathbf{L}}(t) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t) \varphi_k,$$

where $\{\beta_k\} \subset \mathbf{C}^l\mathbf{L}_2$ is a sequence of Brownian motion on $\mathbb{R}_+$ [4, 5].

The following lemma gives the opportunity to transfer all the considerations of Section 1 to the spaces of the random $\mathbf{K}$-values.

Lemma 1. *The operator $A : \mathfrak{U} \to \mathfrak{F}$ is a linear and continuous operator ($A \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$) if and only if the same operator $A : \mathbf{U_K L}_2 \to \mathbf{F_K L}_2$ is a linear and continuous operator ($A \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$).*

Remark 5. Let $\sigma_1^L(M)$ be the L-spectrum of the operator M, where the operators $L,\ M : \mathfrak{U} \to \mathfrak{F}$, and $\sigma_2^L(M)$ be the L-operator spectrum of the operator M, where the operators $L,\ M : \mathbf{U_K L}_2 \to \mathbf{F_K L}_2$. Then $\sigma_1^L(M) = \sigma_2^L(M)$.

Assume that the operators $L,\ M \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$, consider the equation

$$L \overset{o}{\eta} = M\eta + N(\eta). \tag{14}$$

Let $\mathcal{J} = \{0\} \cup \mathbb{R}_+$. A stochastic $\mathbf{K}$-process $\eta \in \mathbf{C}^1(\mathcal{J}; \mathbf{U_K L}_2)$ is called a *solution to equation (14)*, if all its trajectories satisfy equation (14) for all $t \in \mathcal{J}$. A solution $\eta = \eta(t)$ to equation (14) is called a *solution to the Cauchy problem*

$$\lim_{t \to +\infty} (\eta(t) - \eta_0) = 0, \tag{15}$$

if equality (15) holds for some random $\mathbf{K}$-value $\eta_0 \in \mathbf{U_K L}_2$.

Definition 6. *The set $\mathbf{P_K L}_2 \subset \mathbf{U_K L}_2$ is called a stochastic phase space of equation (14), if*

(i) probably almost every solution path $\eta = \eta(t)$ of equation (14) belongs to $\mathbf{P_K L}_2$, i.e. $\eta(t) \in \mathbf{P_K L}_2, t \in \mathbb{R}$, for almost all trajectories;

(ii) for almost all $\eta_0 \in \mathbf{P_K L}_2$, there exists a solution to problem (14), (15).

Let M be the (L, p)-bounded operator. Then we can extend the projector P considered in Section 1 from the Banach space $\mathfrak{U}$ to the space of the random $\mathbf{K}$-values $\mathbf{U_K L}_2$. If condition (11) is satisfied, then we extend the projectors P_l and P_r by $\mathbf{U_K L}_2$. Denote $\mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2 = \mathrm{im} P$, $\mathbf{U}^l_{\mathbf{K}}\mathbf{L}_2 = \mathrm{im} P_l$ and $\mathbf{U}^r_{\mathbf{K}}\mathbf{L}_2 = \mathrm{im} P_r$. Along with semilinear equation (14), we consider the linear equation

$$L \overset{o}{\eta} = M\eta \tag{16}$$

with the initial condition

$$\eta(0) = \eta_0. \tag{17}$$

Theorem 4. *Let the operators* $L,\ M \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$ *and* M *be the* (L,p)*-bounded operator. Then the phase space of equation (16) is the space* $\mathbf{U^1_K L}_2$.

Remark 6. Under the conditions of Theorem 4, if there exists an operator $L^{-1} \in \mathcal{L}(\mathbf{F_K L}_2; \mathbf{U_K L}_2)$, then $\mathbf{U^1_K L}_2 = \mathbf{U_K L}_2$.

Definition 7. *The subspace* $\mathbf{I_K L}_2 \subset \mathbf{U_K L}_2$ *is called the invariant space of equation (16), if, for any* $\eta_0 \in \mathbf{I_K L}_2$*, solution to problem (16), (17)* $\eta \in \mathbf{C}^1(\mathbb{R}; \mathbf{I_K L}_2)$.

Remark 7. If equation (16) has a phase space $\mathbf{P_K L}_2$ and an invariant space $\mathbf{I_K L}_2$, then $\mathbf{I_K L}_2 \subset \mathbf{P_K L}_2$.

Definition 8. *Solutions* $\eta = \eta(t)$ *of equation (16) have an exponential dichotomy if*

(i) the phase space $\mathbf{P_K L}_2$ *of equation (16) splits into a direct sum of two invariant spaces (i.e.* $\mathbf{P_K L}_2 = \mathbf{I^+_K L}_2 \oplus \mathbf{I^-_K L}_2$*);*

(ii) there exist constants $N_k \in \mathbb{R}_+, \nu_k \in \mathbb{R}_+, k = 1, 2$ *such that*

$$\begin{array}{ll} \|\eta^1(t)\|_{\mathbf{U_K L}_2} \leq N_1 e^{-\nu_1(s-t)} \|\eta^1(s)\|_{\mathbf{U_K L}_2} & \text{for } s \geq t, \\ \|\eta^2(t)\|_{\mathbf{U_K L}_2} \leq N_2 e^{-\nu_2(t-s)} \|\eta^2(s)\|_{\mathbf{U_K L}_2} & \text{for } t \geq s, \end{array}$$

where $\eta^1 = \eta^1(t) \in \mathbf{I^+_K L}_2$ *and* $\eta^2 = \eta^2(t) \in \mathbf{I^-_K L}_2$ *for all* $t \in \mathbb{R}$*. The space* $\mathbf{I^+_K L}_2$ $(\mathbf{I^-_K L}_2)$ *is called the stable (unstable) invariant space of equation (16).*

Theorem 5. [9] *Let* M *be the* (L,p)*-bounded operator and condition (11) be fulfilled. Then the solutions of equation (16) have an exponential dichotomy and the spaces* $\mathbf{U^l_K L}_2$ *and* $\mathbf{U^r_K K}_2$ *are stable and unstable invariant spaces of equation (16).*

Next, we arrive at questions about the solvability and stability of stochastic semilinear equation (3). If, for some fixed $\omega \in \Omega$, there exists a solution $\eta = \eta(t)$ to equation (14), then η belongs to the set

$$\mathbf{M_K L}_2 = \begin{cases} \{\eta \in \mathbf{U_K L}_2 : (\mathbb{I} - Q)(M\eta + N(\eta)) = 0\}, \text{ if } \ker L \neq \{0\}; \\ \mathbf{U_K L}_2, \text{ if } \ker L = \{0\}, \end{cases}$$

and the following theorem is true.

Theorem 6. [8] *Let* M *be the* (L,p)*-bounded operator, the operator* $N \in C^1(\mathbf{U_K L}_2, \mathbf{F_K L}_2)$*, and the set* $\mathbf{M_K L}_2$ *be a simple Banach* C^1*-manifold at the point* $\eta_0 \in \mathbf{U_K L}_2$*. Then the set* $\mathbf{M_K L}_2$ *is the phase space of equation (14).*

Definition 9. *The set*

$$\mathbf{M^{+(-)}_K L}_2 = \{\eta_0 \in \mathbf{U_K L}_2 : \|P_{l(r)}\eta_0\|_{\mathbf{U_K L}_2} \leq R_1,\ \|\eta(t,\eta_0)\|_{\mathbf{U_L L}_2} \leq R_2,\ t \in \mathbb{R}_{+(-)}\}$$

is such that

(i) $\mathbf{M^{+(-)}_K L}_2$ *is diffeomorphic to a closed ball in* $\mathbf{I^{+(-)}_K L}_2$*;*

(ii) $\mathbf{M^{+(-)}_K L}_2$ *concerns* $\mathbf{I^{+(-)}_K L}_2$ *at the zero point;*

(iii) for any $\eta_0 \in \mathbf{M^{+(-)}_L L}_2$*, for* $t \to +(-)\infty$*,* $\|\eta(t,\eta_0)\|_{\mathbf{U_L L}_2} \to 0$ *is called a stable (unstable) invariant manifold of equation (14).*

From Lemma 1 and Theorem 3 we obtain the following result.

Theorem 7. *Let M be the (L,p)-bounded operator, condition (11) be fulfilled, and the operator $N \in C^k(\mathfrak{U},\mathfrak{F})$ be such that $N(0) = 0$, $N'_0 = \mathbb{O}$. Then there exist stable and unstable invariant manifolds of equation (14) in the neighborhood of the zero point.*

3. Hoff Stochastic Equation

Consider the stochastic analogue of equation (1). Let $\mathfrak{U} = \overset{\circ}{W}{}_2^1$, $\mathfrak{F} = W_2^{-1}$ (functional spaces are defined on the domain Σ). The space $\mathfrak{U}$ is a real separable Hilbert space densely and continuously nested in $\mathfrak{F}$. In the space $\mathfrak{U}$, basis is orthonormal in the sense of $\mathfrak{U}$ of consecutive eigenfunctions $\{\varphi_k\}$ of the Laplace operator Δ corresponding to $\{\nu_k\}$. Here $\{\nu_k\}$ is a sequence of eigenvalues of the Laplace operator numbered in nondecreasing order taking into account multiplicity. By analogy with Section 2, we construct the spaces of the random $\mathbf{K}$-values $\mathbf{U_K L}_2$, $\mathbf{F_K L}_2$ and the spaces of differentiable "noise" $\mathbf{C}^l\mathbf{U_K L}_2$, $l \in \{0\} \bigcup \mathbb{N}$. Let $\mathbf{K} = \{\lambda_k\}$ be a sequence such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. For example (see [5]), as $\mathbf{K} = \{\lambda_k\}$, we can choose a sequence of eigenvalues of the Green operator $\lambda_k = |\nu_k|^{-m}$ (here $m \in \mathbb{N}$ is chosen in such a way that the series $\sum\limits_{k=1}^{\infty} |\nu_k|^{-m}$ converges).

The operators L, M and N are defined by formulas

$$L : \chi \to (\lambda + \Delta)\chi, \chi \in \mathbf{U}_{W\mathbf{K}}\mathbf{L}_2, \ M : \chi \to \alpha\Delta\chi, \ N : \eta \to \beta\chi^3, \ \chi \in \mathbf{U_K L}_2. \tag{18}$$

Then the stochastic analogue of Hoff equation (1) is represented as the equation

$$L \overset{o}{\chi} = M\chi + N(\chi). \tag{19}$$

Lemma 2. *For any $\lambda \in \mathbb{R}_+$ and $\alpha, \ \beta \in \mathbb{R} \setminus \{0\}$,*

(i) the operators $L, \ M \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$;

(ii) the operator M is $(L,0)$-bounded operator;

(iii) the operator $N \in C^\infty(\mathbf{U_K L}_2; \mathbf{U_K L}_2)$, $N(0) = 0$ and $N'_0 \equiv \mathbb{O}$.

Proof. (i) According to Lemma 1, the operators $L, \ M \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$.

(ii) The L-spectrum of the operator M has the form

$$\sigma^L(M) = \left\{ \mu_k = \frac{\alpha}{\lambda + \nu_k}, \quad \nu_k \neq -\lambda \right\}, \tag{20}$$

therefore it is bounded. The kernel of the operator L has the form

$$\ker L = \operatorname{span}\{\varphi_l : \nu_l = -\lambda\}.$$

If $\psi \in \ker L \setminus \{0\}$ then

$$\psi = \sum_{\nu_l = -\lambda} a_l \varphi_l, \quad \sum_{\nu_l = -\lambda} |a_l| > 0$$

and

$$M\psi = \alpha \sum_{\nu_l = -\lambda} a_l \varphi_l \notin \operatorname{im} L.$$

(iii) Let $\chi \in \mathbf{U_K L_2}$. Then the Frechet derivatives N'_η, $N"_\chi$ of the operator N at the point η has the form

$$N'_\chi : \chi \to 3\beta\chi^2, \ N''_\chi : \chi \to 6\beta\chi.$$

All other Frechet derivatives of the operator N at the point u are zero. Obviously, the operator $N(0) = 0$ and the Frechet derivative $N'_0 = \mathbb{O}$.

□

Theorem 8. *Let $\alpha\beta > 0$. Then the phase space of equation (19) is the set*

$$\mathbf{M_K L_2} == \begin{cases} \{\chi \in \mathbf{U_K L_2} : \sum\limits_{-\lambda=\nu_l} < (1-\alpha-\beta\chi^2)\chi, \varphi_l > \varphi_l = 0, \ if \ -\lambda = \nu_l; \\ \mathbf{U_K L_2}, \ if \ \lambda \neq \nu_j. \end{cases}$$

Proof. The statements of the theorem follow from Lemma 1, Lemma 19, Theorem 6 and the results of [1].

□

Remark 8. The subspace $\mathbf{U^1_K L_2} = \{\chi \in \mathbf{U_K L_2} :< \chi, \varphi_l >= 0, -\lambda = \nu_l\}$ is a model of the manifold $\mathbf{M_K L_2}$ for $-\lambda = \nu_l$.

Theorem 9. *Let $\alpha, \ \beta, \ \lambda \in \mathbb{R}_+$.*

(i) If $\lambda \leq -\nu_1$, then equation (19) has only a stable invariant manifold that coincides with $\mathbf{M_K L_2}$.

(ii) If $-\nu_1 < \lambda$, then there exist a finite-dimensional unstable invariant manifold $\mathbf{M^+_K L_2}$ and an infinite-dimensional stable invariant manifold $\mathbf{M^-_K L_2}$ of equation (19) in the neighborhood of the zero point.

Proof. (i) Let $\lambda \leq \nu_1$, then the L-spectrum of the operator M has the form

$$\sigma^L(M) = \sigma^L_-(M) = \left\{ \frac{\alpha}{\lambda + \nu_k} : \lambda < -\nu_1 \right\}.$$

Following Theorem 5, the linear part of equation (19) has only a stable invariant space that coincides with the subspace $\mathbf{U^1_K L_2}$. The existence of an unstable invariant manifold that coincides with $\mathbf{M_K L_2}$ follows from Theorem 7 and Theorem 8.

(ii) If $\lambda > -\nu_1$, then the L-spectrum of the operator M consists of two parts $\sigma^L(M) = \sigma^L_-(M) \bigcup \sigma^L_+(M)$, where

$$\sigma^L_-(M) = \left\{ \frac{\alpha}{\lambda + \nu_k} : \lambda < -\nu_k \right\}, \ \sigma^L_r(M) = \left\{ \frac{\alpha}{\lambda + \nu_k} : \lambda > -\nu_k \right\}.$$

The existence of a stable manifold $\mathbf{M^+_K L_2}$ of equation (19) follows from Theorem 5, Theorem 7 and Theorem 8. The model of the set $\mathbf{M^+_K L_2}$ is an infinite-dimensional subspace $\mathbf{I^+_K L_2} = \overline{span}\{\nu_l : -\lambda \leq \nu_l\}$. The finite-dimensional space $\mathbf{I^-_K L_2} = span\{\nu_l : -\nu_l < \lambda\}$ is a model of an unstable manifold $\mathbf{M^-_K L_2}$.

□

Conclusion

In the future, following [12–14], it is proposed to conduct numerical experiments on the study of invariant manifolds of the stochastic analogue of the Hoff model.

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ИНВАРИАНТНЫЕ МНОГООБРАЗИЯ МОДЕЛИ ХОФФА В ПРОСТРАНСТВАХ «ШУМОВ»

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В данной работе изучается стохастический аналог уравнения Хоффа, который является моделью отклонения двутавровой балки от положения равновесия. Показана устойчивость модели при некоторых значениях параметров данной модели. При исследовании модель рассматривается как стохастическое полулинейное уравнение соболевского типа, где стохастический процесс выступает в качестве искомой величины. Установлены достаточные условия существования инвариантных многообразий полулинейного стохастического уравнения соболевского типа. Полученные результаты перенесены на уравнение Хоффа, рассматриваемого в специально построенных пространствах «шумов». Доказано, что в окрестности точки нуль существуют конечномерное неустойчивое и бесконечномерное устойчивое инвариантные многообразия уравнения Хоффа при положительных значениях параметров, которые определяют свойства материала балки и нагрузку на балку.

Ключевые слова: производная Нельсона – Гликлиха; стохастические уравнения соболевского типа; инвариантные многообразия.

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