КРАТКИЕ СООБЩЕНИЯ

MSC 93E10 DOI: 10.14529/mmp210408

NON-UNIQUENESS OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS WITH WENTZELL CONDITION

N.S. Goncharov[1]***, S.A. Zagrebina***[1]***, G.A. Sviridyuk***[1]
[1]South Ural State University, Chelyabinsk, Russian Federation
E-mails: goncharovns@susu.ru, zagrebinasa@susu.ru, sviridiukga@susu.ru

Recently, in the mathematical literature, the Wentzell boundary condition is considered from two points of view. In the first case, let us call it classical one, this condition is an equation containing a linear combination of the values of the function and its derivatives on the boundary of the domain. Moreover, the function itself also satisfies the equation with an elliptic operator defined in the domain. In the second case, which we call neoclassical one, the Wentzell condition is an equation with the Laplace–Beltrami operator defined on the boundary of the domain understood as a smooth compact Riemannian manifold without boundary, and the external action is represented by the normal derivative of a function defined in the domain. The paper shows the non-uniqueness of solutions to boundary value problems with the Wentzell condition in the neoclassical sense both for the equation with the Laplacian and for the equation with the Bi-Laplacian given in the domain.

Keywords: Wentzell condition.

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$ be a bounded connected domain with the boundary $\partial\Omega$ of the class C^∞. For the first time, the Wentzell boundary condition

$$\Delta u(x) + \alpha \frac{\partial u}{\partial \nu}(x) + \beta u(x) = 0, \ x \in \partial\Omega, \tag{1}$$

appeared in [1]. Boundary value problems with condition (1) for second-order linear elliptic equations were studied by various methods [2–6]. Over time [7], condition (1) was understood as a description of a process occurring at the boundary of the domain and influenced by processes within the domain. The work [8] was the first to represent condition (1) in the following form

$$\Delta u_2(x) + \alpha \frac{\partial u_1}{\partial \nu}(x) + \beta u_2(x) = 0, \ x \in \partial\Omega. \tag{2}$$

Here the boundary $\partial\Omega$ is understood as a compact smooth Riemannian manifold without boundary, Δ is the Laplace–Beltrami operator, and the second term characterizes the influence of the processes occurring inside the domain.

In this context, consider condition (2) together with the Laplacian

$$\Delta u_1(x) = 0, \ x \in \Omega. \tag{3}$$

By solution to problem (2), (3) we mean the function

$$u(x) = \begin{cases} u_1(x), \ x \in \Omega; \\ u_2(x), \ x \in \partial\Omega. \end{cases} \tag{4}$$

Perform the replacemnet

$$\frac{\partial u_1}{\partial \nu}(x) = \varphi(x),\ x \in \partial\Omega. \tag{5}$$

Following [9], we can always find a pair of Banach spaces in $\mathfrak{U}_1 = \mathfrak{U}_1(\Omega)$ and $\mathfrak{F} = \mathfrak{F}(\partial\Omega)$ such that for any function $\varphi \in \mathfrak{F}$ there exists the unique solution $u_1 \in \mathfrak{U}_1$ to problem (3), (5). Following [10], we can find the Banach space $\mathfrak{U}_2 = \mathfrak{U}_2(\partial\Omega)$ for the space $\mathfrak{F}$ and find the coefficients $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that there exists a unique solution $u_2 \in \mathfrak{U}_2$ to problem (2), (5) for any function $\varphi \in \mathfrak{F}$. Obviously, due to the arbitrary choice of φ, solution (4) to problem (2), (3) cannot be unique.

A similar situation arises if we replace a Laplacian with the Bi-Laplacian

$$\Delta^2 u_1(x) = 0,\ x \in \Omega. \tag{6}$$

For completeness, we introduce the Dirichlet condition

$$u_1(x) = 0,\ x \in \partial\Omega. \tag{7}$$

Reasoning by analogy with the previous case, we find a triple of Banach spaces $\mathfrak{U}_1$, $\mathfrak{U}_2$ and $\mathfrak{F}$ such that for any $\varphi \in \mathfrak{F}$ there exist unique solutions to problems (5)–(7) and (3), (5). However, the solution to problem (2), (6), (7) cannot be unique.

Also, note that despite all the above, under boundary conditions (7) and

$$\Delta u_1(x) + \alpha\frac{\partial u_1}{\partial \nu}(x) + \beta u_1(x) = 0,\ x \in \partial\Omega,$$

the solution to equation (6) exists and is unique in a suitably chosen space [11].

Acknowledgements. *The research was funded by RFBR and Chelyabinsk Region, project number 20-41-740010.*

References

1. Ventcel' A.D. On Boundary Conditions for Multidimensional Diffusion Processes. *Theory of Probability and Its Applications*, 1960, vol. 4, no. 2, pp. 164–177.
2. Luo Y., Trudinger N.S. Linear Second Order Elliptic Equations with Venttsel Boundary Conditions. *Proceedings of the Royal Society of Edinburgh. Section A: Mathematics*, 1991, vol. 118, no. 3–4, pp. 193–207.
3. Apushkinskaya D.E., Nazarov A.I. An Initial-Boundary Value Problem with a Venttsel' Boundary Condition for Parabolic Equations not in Divergence Form. *St. Petersburg Mathematical Journal*, 1995, vol. 6, no. 6, pp. 1127–1149.
4. Lukyanov V.V., Nazarov A.I. Solving of Vent'sel Boundary-Value Problem for Laplace and Helmholtz Equations by Iterated Potentials. *Journal of Mathematical Sciences*, 2000, vol. 102, no. 4, pp. 4265–4274.
5. Favini A., Goldstein G.R., Goldstein J.A., Romanelli S. C_0-Semigroups Generated by Second Order Differential Operators with General Wentzell Boundary Conditions. *Proceedings of the American Mathematical Society*, 2000, vol. 128, no. 7, pp. 1981–1989.
6. Favini A., Goldstein G.R., Goldstein J.A., Romanelli S. The Heat Equation with Generalized Wentzell Boundary Condition. *Journal of Evolution Equations*, 2002, vol. 2, no. 1, pp. 1–19.

7. Goldstein G.R. Derivation and Physimathcal Interpretation of General Boundary Conditions. *Advances in Differential Equations*, 2006, vol. 4, no. 11, pp. 419–456.
8. Denk R., Kunze M., Ploss D. The Bi-Laplacian with Wentzell Boundary Conditions on Lipschitz Domains. *Integral Equations and Operator Theory*, 2021, vol. 93, no. 2, p. 13.
9. Triebel H. *Interpolation Theory. Function Spaces. Differential Operators.* Berlin, Veb Deutscher Verlag der Wissenschaften, 1978.
10. Warner F.W. *Foundations of Differentiable Manifold and Lie Groups.* New York, Berlin, Heidelberg, Tokyo, Springer, 1983.
11. Goncharov N.S., Zagrebina S.A., Sviridyuk G.A. Showalter–Sidorov and Cauchy Problems for the Linear Dzektser Equation with Wentzel and Robin Boundary Conditions in a Bounded Domain. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2022. (in print)

Received August 3, 2021

УДК 517.9 **DOI: 10.14529/mmp210408**

НЕЕДИНСТВЕННОСТЬ РЕШЕНИЙ КРАЕВЫХ ЗАДАЧ С УСЛОВИЕМ ВЕНТЦЕЛЯ

Н.С. Гончаров[1]***, С.А. Загребина***[1]***, Г.А. Свиридюк***[1]
[1]Южно-Уральский государственный университет, г. Челябинск, Российская Федерация

В последнее время в математической литературе краевое условие Вентцеля рассматривается с двух точек зрения. В первом случае, назовем его классическим, это условие представляет собой уравнение, содержащее линейную комбинацию значений функции и ее производных на границе области. Причем сама функция удовлетворяет еще уравнению с эллиптическим оператором, заданным в области. Во втором, неоклассическом случае условие Вентцеля представляет собой уравнение с оператором Лапласа – Бельтрами, заданным на границе области, понимаемой как гладкое компактное риманово многообразие без края, причем внешнее воздействие представлено нормальной производной функции, заданной в области. В заметке показана неединственность решений краевых задач с условием Вентцеля в неоклассическом смысле как для уравнения с лапласианом, так и для уравнения с билапласианом, заданными в области.

Ключевые слова: условие Вентцеля.

Литература

1. Вентцель, А.Д. О граничных условиях для многомерных диффузионных процессов / А.Д. Вентцель // Теория вероятней и ее применения. – 1959. – Т. 4, № 2. – С. 172–185.
2. Luo, Y. Linear Second Order Elliptic Equations with Venttsel Boundary Conditions / Y. Luo, N.S. Trudinger // Proceedings of the Royal Society of Edinburgh. Section A: Mathematics. – 1991. – V. 118, № 3–4. – P. 193–207.
3. Апушинская, Д.Е. Начально-краевая задача с граничным условием Вентцеля для недивергентных параболических уравнений / Д.Е. Апушинская, А.И. Назаров // Алгебра и анализ. – 1994. – Т. 6, № 6. – С. 1–29.

4. Лукьянов, В.В. Решение задачи Вентцеля для уравнения Лапласа и Гельмгольца с помощью повторных потенциалов / В.В. Лукьянов, А.И. Назаров // записки научных семинаров Санкт-Петербургского отделения математического института им. В.А. Стеклова РАН. – 1998. – № 250. – С. 203–218.
5. Favini, A. C_0-Semigroups Generated by Second Order Differential Operators with General Wentzell Boundary Conditions / A. Favini, G.R. Goldstein, J.A. Goldstein, S. Romanelli // Proceedings of the American Mathematical Society. – 2000. – V. 128, № 7. – P. 1981–1989.
6. Favini, A. The Heat Equation with Generalized Wentzell Boundary Condition / A. Favini, G.R. Goldstein, J.A. Goldstein, S. Romanelli // Journal of Evolution Equations. – 2002. – V. 2, № 1. – P. 1–19.
7. Goldstein, G.R. Derivation and Physical Interpretation of General Boundary Conditions / G.R. Goldstein // Advances in Differential Equations. – 2006. – V. 4, № 11. – P. 419–456.
8. Denk, R. The Bi-Laplacian with Wentzell Boundary Conditions on Lipschitz Domains / R. Denk, M. Kunze, D. Ploss // Integral Equations and Operator Theory. – 2021. – V. 93, № 2. – P. 13.
9. Triebel, H. Interpolation Theory. Function Spaces. Differential operators / H. Triebel. – Berlin: Veb Deutscher Verlag der Wissenschaften, 1978.
10. Warner, F.W. Foundations of Differentiable Manifold and Lie Groups / F.W. Warner. – New York, Berlin, Heidelberg, Tokyo: Springer, 1983.
11. Гончаров, Н.С. Задачи Шоуолтера–Сидорова и Коши для линейного уравнения Дзекцера с краевыми условиями Вентцеля и Робена в ограниченной области / Н.С. Гончаров, С.А. Загребина, Г.А. Свиридюк // Вестник ЮУрГУ. Серия: Математика. Механика. Физика. – 2022. (в печати)

Никита Сергеевич Гончаров, аспирант, кафедра уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), goncharovns@susu.ru.

Софья Александровна Загребина, доктор физико-математических наук, профессор, кафедра математического и компьютерного моделирования, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), zagrebinasa@susu.ru.

Георгий Анатольевич Свиридюк, доктор физико-математических наук, профессор, научно-исследовательская лаборатория неклассических уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), sviridiukga@susu.ru.

Поступила в редакцию 3 августа 2021 г.