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SEMILINEAR SOBOLEV TYPE MATHEMATICAL MODELS

***A.A. Zamyshlyaeva*[1], *E.V. Bychkov*[1]**
[1]South Ural State University, Chelyabinsk, Russian Federation
E-mail: zamyshliaevaaa@susu.ru, bychkovev@susu.ru

The article contains a review of the results obtained in the scientific school of Georgy Sviridyuk in the field of semilinear Sobolev type mathematical models. The paper presents results on solvability of the Cauchy and Showalter–Sidorov problems for semilinear Sobolev type equations of the first, the second and higher orders, as well as examples of non-classical models of mathematical physics, such as the generalized Oskolkov model of nonlinear filtering, propagation of ion-acoustic waves in plasma, propagation waves in shallow water, which are studied by reduction to one of the above abstract problems. Methods for studying the semilinear Sobolev type equations are based on the theory of relatively p-bounded operators for equations of the first order and the theory of relatively polynomially bounded operator pencils for equations of the second and higher orders in the variable t. The paper uses the phase space method, which consists in reducing a singular equation to a regular one defined on some subspace of the original space, to prove existence and uniqueness theorems, and the Galerkin method to construct an approximate solution.

*Keywords: Oskolkov equation; equation of ion-acoustic waves in plasma; modified Boussinesq equation; semilinear Sobolev type equation; relatively p-bounded operators; relatively polynomially bounded operator pencils; Galerkin method; *-weak convergence.*

Dedicated to the 70th anniversary of the Teacher
Professor Georgy Anatolyevich Sviridyuk

Introduction

Mathematical models based on semilinear Sobolev type equations are called semilinear Sobolev type mathematical models.

The necessity to study semilinear Sobolev type mathematical models caused by the need to study applied problems related to the dynamics of viscoelastic fluids [16], vibrations in the DNA molecule [6], the theory of metal creep [5], wave propagation in shallow water [25], the propagation of ion-acoustic waves in plasma [1], the theory of electric circuits [14], the theory of heat conduction with two temperatures [4], filtration in a fractured-porous medium [2, 28] and others [19, 39].

The study of semilinear Sobolev type equations was initiated in [29,37] and the concept of a quasistationary trajectory was introduced. Later, on the basis of abstract results, the Oskolkov [33, 35] and Hoff [32] mathematical models were investigated, and the structure of the phase space of the Hoff and Oskolkov equations was studied. In parallel with the solution of the problem of existence and uniqueness, the theory of optimal control of solutions to the semilinear Sobolev type equations arose and was developed [23, 36, 44], the theory of stability of solutions to the semilinear Sobolev type equations [22, 34] was studied and the phenomenon of non-uniqueness of solutions was explained [18, 31].

Probably, the first work devoted to the study of equations unsolvable with respect to the highest time derivative belongs to A. Poincare (1885) [20]. However, the regular study of initial-boundary value problems for such equations began with the works of

S.L. Sobolev [27]. At present, the theory of Sobolev type equations is actively developing both in breadth and in depth, as evidenced by many scientific directions around which the scientific schools have developed [1, 7, 10, 14, 15, 21, 24, 26]. In this paper, we trace the stages in the development of the theory of semilinear Sobolev type equations and consider its applications to the study of three semilinear Sobolev type mathematical models.

The first one is the Oskolkov mathematical model (a mathematical model of nonlinear filtration in a fractured-porous medium). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^∞. In a cylinder $\Omega \times \mathbb{R}$, consider the equation

$$u_t - æ\Delta u_t = \nu\Delta u - K(u) \tag{1}$$

with the Cauchy–Dirichlet conditions

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3}$$

Equation (1) describes many processes and phenomena occurring with the participation of a viscoelastic fluid, including filtration. It was obtained by A.P. Oskolkov [19]. The nonlinear term in (1) is such that $K(0) = 0$, $\langle K(u), u\rangle \geq 0$ ($\langle\cdot,\cdot\rangle$ is an inner product in $L^2(\Omega)$). In particular, it can take the form $K(u) = u^{2m+1}$ or $K(u) = \operatorname{sh} u$. Generally speaking, the nonlinearity can be represented by the series

$$K(u) = \sum_{m=0}^{\infty} a_m u^{2m+1}, \quad a_m \in \overline{\mathbb{R}}_+.$$

The parameters $æ, \nu \in \mathbb{R}_+$ characterize the elastic and viscous properties of the fluid, respectively.

The second one is a mathematical model of ion-acoustic waves in plasma. Let $\Omega = (0,a) \times (0,b) \times (0,c) \subset \mathbb{R}^3$. In a cylinder $\Omega \times \mathbb{R}$ consider the equation

$$(\Delta - \lambda)u_{tttt} + (\Delta - \lambda')u_{tt} + \alpha\frac{\partial^2 u}{\partial x_3^2} = \Delta(u^3) \tag{4}$$

with the Cauchy–Dirichlet conditions

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ u_{tt}(x,0) = u_2(x), \quad u_{ttt}(x,0) = u_3(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \end{gathered} \tag{5}$$

Equation (4) describes the ion-acoustic waves in a plasma in an external magnetic field. Here function u is a generalized potential of the electric field, the constants λ, λ', α characterize ion gyrofrequency, Langmuir frequency and Debye radius.

The third one is a mathematical model of wave propagation in shallow water. It is based on a modified Boussinesq equation. Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary $\partial\Omega$ of class C^∞, $T \in \mathbb{R}_+$. In a cylinder $C = \Omega \times (0,T)$, consider the modified Boussinesq equation

$$(\lambda - \Delta)u_{tt} - \alpha^2\Delta u + u^3 = 0, \quad (x,t) \in \Omega \times (0,T) \tag{6}$$

with the Cauchy–Dirichlet conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \tag{7}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{8}$$

where $\lambda, \alpha \in \mathbb{R}$. Other modifications of the Boussinesq equation with nonlinearities of the form $\Delta|u|^p u$ have also become widespread [4, 5, 17]. The equation has many applications in various fields of natural science. For example, it models the propagation of waves in shallow water, taking into account capillary effects. In this case, the function $u = u(x,t)$ determines the height of the wave.

The article, in addition to the Introduction, Conclusion and References, includes seven paragraphs. The first section contains the main results of the theory of p-bounded operators obtained by G.A. Sviridyuk, necessary for further presentation. The second section presents results on the solvability of the Cauchy problem for semilinear first-order equations obtained by V.O. Kazak. The third paragraph contains some results of the study of Oskolkov generalized mathematical model. The fourth section presents the main statements of the theory of relatively polynomially bounded operator pencils, obtained by A.A. Zamyshlyaeva. The fifth section contains results on the solvability of the Cauchy problem for high-order semilinear Sobolev-type equations obtained in the authors' papers. The sixth paragraph is devoted to the study of the semilinear model of ion-acoustic waves. The seventh paragraph contains the results of the study of the mathematical model of shallow water wave propagation.

1. Relatively p-Bounded Operators

A detailed exposition of the theory of relatively p-bounded operators can be found in [30]. Let $\mathfrak{U}$, $\mathfrak{F}$ be Banach spaces and operators L, $M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$.

Definition 1. The set

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$$

is called a *resolvent* set of the operator M with respect to operator L (in short, L-*resolvent set* of the operator M). The set $\mathbb{C}\backslash\rho^L(M) = \sigma^L(M)$ is called a *spectrum* of the operator M with respect to the operator L (in short, the L-*spectrum* of the operator M).

Definition 2. Operator-functions

$$(\mu L - M)^{-1}, \quad R^L_\mu = (\mu L - M)^{-1}L, \quad L^L_\mu = L(\mu L - M)^{-1}$$

with domain $\rho^L(M)$ are called respectively *resolvent*, *right resolvent*, *left resolvent* of the operator M with respect to the operator L (in short, the *L-resolvent, right L-resolvent, left L-resolvent of the operator M*).

Theorem 1. [30] *Let the operator $L \in \mathcal{L}(\mathfrak{U},\mathfrak{F})$, and the operator M :* dom $M \subset \mathfrak{U} \to \mathfrak{F}$ *be linear and closed. Then the L-resolvent, the right and the left L-resolvent are analytic in the set $\rho^L(M)$.*

Definition 3. The operator M is said to be spectrally bounded with respect to the operator L (in short, (L,σ)-*bounded*), if

$$\exists a > 0 \ \forall \mu \in \mathbb{C} : (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Lemma 1. [30] *Let the operator M be (L,σ)-bounded. Then the operators*

$$P=\frac{1}{2\pi i}\int\limits_{\Gamma}R^L_\lambda(M)d\lambda \text{ and } Q=\frac{1}{2\pi i}\int\limits_{\Gamma}L^L_\lambda(M)d\lambda$$

are projectors. Here $\Gamma=\{\lambda\in\mathbb{C}:|\lambda|=r>a\}$.

Set $\mathfrak{U}^0=\ker P$, $\mathfrak{F}^0=\ker Q$, $\mathfrak{U}^1=\operatorname{im}P$, $\mathfrak{F}^1=\operatorname{im}Q$. Denote by $L_k(M_k)$ the restriction of the operator L (M) to the subspace $\mathfrak{U}^k$, $k=0,1$.

Theorem 2. [30] *Let the operator M be (L,σ)-bounded. Then*
(i) $L_k,M_k:\mathfrak{U}^k\to\mathfrak{F}^k, k=0,1$;
(ii) $M_0^{-1}\in\mathcal{L}(\mathfrak{F}^0,\mathfrak{U}^0)$;
(iii) operator $L_1^{-1}\in\mathcal{L}(\mathfrak{F}^1,\mathfrak{U}^1)$ exists;
(iv) operator $M_1\in\mathcal{L}(\mathfrak{U}^1,\mathfrak{F}^1)$ exists.

Let $\varphi_0\in\ker L\backslash\{0\}$ be an eigenvector of the operator L.

Definition 4. *An ordered set $\{\varphi_1,\varphi_2,\ldots\}\subset\operatorname{im}L$ is called a chain of M-adjoined vectors of an eigenvector φ_0 if*

$$L\varphi_{q+1}=M\varphi_q,\ q=0,1,2\ldots,\quad \varphi_q\notin\ker L \textit{ for } q=1,2,\ldots$$

The chain is said to be finite if there exists an M-adjoined vector φ_p such that either $\varphi_p\notin\operatorname{dom}M$ or $M\varphi_p\notin\operatorname{im}L$. The power of the final chain is called its length. The linear span of all eigenvectors and M-adjoined vectors of the operator L is called the M-root lineal of the operator L. Under the conditions of the theorem 2, we construct the operators $H=M_0^{-1}L_0\in\mathcal{L}(\mathfrak{U}^0)$ and $S=L_1^{-1}M_1\in\mathcal{L}(\mathfrak{U}^1)$. Since the operator-function $(\mu L_0-M_0)^{-1}$ is an entire function, it can therefore be expanded into a Taylor series

$$(\mu L_0-M_0)^{-1}=(\mu H-\mathbb{I})^{-1}M_0^{-1}=\left(-\sum_{k=0}^{\infty}\mu^kH^k\right)M_0^{-1},$$

absolutely and uniformly convergent on any compact set in $\mathbb{C}$. Let's do the same with the operator-function $(\mu L_1-M_1)^{-1}$.

$$(\mu L_1-M_1)^{-1}=(\mu\mathbb{I}-S)^{-1}L_1^{-1}=\mu^{-1}(\mathbb{I}-\mu^{-1}S)^{-1}L_1^{-1}=$$

$$=\mu^{-1}\left(\sum_{k=0}^{\infty}\mu^{-k}S^k\right)L_1^{-1},$$

where $\mu\in\rho(S)$ or what is the same as $\mu\in\rho^L(M)$. Hence, for the (L,σ)-bounded operator, by virtue of the last two expansions, we have

$$(\mu L-M)^{-1}=\left(-\sum_{k=0}^{\infty}\mu^kH^k\right)M_0^{-1}(\mathbb{I}-Q)+\sum_{k=1}^{\infty}\mu^{-k}S^{k-1}L_1^{-1}Q. \tag{9}$$

Let the operator M be (L,σ)-bounded.

Definition 5. *The infinity point of the L-resolvent of the operator M is called*

- *a removable singular point if $H \equiv \mathbb{O}$;*
- *a pole of order p if $H^p \neq \mathbb{O}, H^{p+1} \equiv \mathbb{O}$, $p \in \mathbb{N}$;*
- *an essentially singular point if $H^q \neq \mathbb{O}$, $\forall q \in \mathbb{N}$.*

Further the removable singular point will be called *a pole of order zero.*

Remark 1. In what follows the (L, σ)-bounded operator M will be called (L, p)-bounded, if the point ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of its L-resolvent.

Theorem 3. [30] *Let L be a Fredholm operator (that is,* $\operatorname{ind} L = 0$*). Then the following statements are equivalent:*

1. *The operator M is $(L, 0)$-bounded;*
2. *Any eigenvector of the operator L does not have M-adjoined vectors.*

2. Semilinear Sobolev Type Equations of the First Order

Let the operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, $N \in C^k(\mathfrak{U}; \mathfrak{F})$, $k \in \mathbb{N} \cup \{\infty\}$, and the operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Consider the Cauchy problem

$$u(0) = u_0 \tag{10}$$

for a semilinear Sobolev type equation

$$L\dot{u} = Mu + N(u). \tag{11}$$

The vector-function $u \in C^k((-T, T); \mathfrak{U})$ is called *a solution to equation* (11) if for some $T \in \mathbb{R}_+$ it satisfies this equation. The solution $u = u(t)$ of equation (11) is called *a solution to problem* (10), (11) if it satisfies the initial condition (10).

Example 1. Let $\mathfrak{U} \equiv \mathfrak{F} \equiv \mathbb{R}^2_{(\xi,\eta)}$, the operators L, M, and N be defined by formulas

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M = \mathbb{I}, \quad N : u \to \begin{pmatrix} 0 \\ -\xi^2 \end{pmatrix}, \quad u = (\xi, \eta).$$

Then the Cauchy problem with $u_0 = (0, 0)$, for equation (11) will have two solutions $(0, 0)$ and $(t/2, t^2/4)$. If instead of the operator N in this case we take the operator $N : u \to \begin{pmatrix} 1 \\ -\xi^2 \end{pmatrix}$, then the same problem will not have a solution at all. This simple example shows the need to narrow down the definition of a solution to equations (11).

By virtue of theorem 2 equation (11) can be reduced to an equivalent system

$$H\dot{u}^0 = u^0 + M_0^{-1}(\mathbb{I} - Q)N(u), \tag{12}$$

$$\dot{u}^1 = Su^1 + L_1^{-1}QN(u), \tag{13}$$

where $u^1 = Pu$, $u^0 = u - u^1$.

Definition 6. Solution $u = u(t)$ of problem (10), (11) is called a *quasi-stationary trajectory* for equation (11) passing through the point u_0, if $H\dot{u}(t) \equiv 0$ for all $t \in (-T, T)$.

Obviously, any stationary solution of problem (10), (11) is a quasi-stationary trajectory, however the opposite is not true. In example 1 a stationary solution is the only quasi-stationary trajectory passing through the point $(0, 0)$, and in this sense such solution is unique. Further only quasi-stationary trajectories are considered.

To this end, introduce a set which is called the phase space of the equation (11):

$$\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(Mu + N(u)) = 0\}.$$

By virtue of theorem 2 and (12) any quasi-stationary trajectory $u = u(t)$ lies in $\mathfrak{M}$, that is $u(t) \in \mathfrak{M}$ for all $t \in (-T, T)$.

Let the point $u_0 \in \mathfrak{M}$, put $u_0^1 = Pu_0 \in \mathfrak{U}^1$. The set $\mathfrak{M}$ at the point u_0 is *a Banach C^k-manifold* if there are neighborhoods $\mathfrak{O}_0^{\mathfrak{M}} \subset \mathfrak{M}$ and $\mathfrak{O}_0^1 \subset \mathfrak{U}^1$ of the points u_0 and u_0^1, respectively, and a C^k-diffeomorphism $\delta : \mathfrak{O}_0^1 \to \mathfrak{O}_0^{\mathfrak{M}}$ such that δ^{-1} is equal to the contraction of P onto $\mathfrak{O}_0^{\mathfrak{M}}$. The set $\mathfrak{M}$ is called a *Banach C^k-manifold modelled by the space* $\mathfrak{U}^1$*,* if it is a Banach C^k-manifold at each point of $\mathfrak{U}^1$.

Theorem 4. [29] *Let the set $\mathfrak{M}$ be a Banach C^k-manifold at the point u_0. Then there is a unique quasi-stationary trajectory of equation (11) passing through the point u_0.*

3. Mathematical Model of Oskolkov

Reduce problem (1) – (3) to the Cauchy problem (10) for the semilinear Sobolev type equation (11). To do this, put $\mathfrak{U} = W_2^{m+2} \cap \overset{\circ}{W}{}_2^1$, $\mathfrak{F} = W_2^m$, $m \in \mathbb{N}$. All functional spaces are defined on the domain Ω. The operators L and M are defined by the formulas $L = 1 - æ\Delta$, $M = \nu\Delta$. Obviously, $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, and L is a Fredholm operator for all $æ \in \mathbb{R} \setminus \{0\}$.

Lemma 2. [33] *For all $æ, \nu \in \mathbb{R} \setminus \{0\}$ the operator M is $(L, 0)$-bounded.*

Denote by $\{\lambda_k\}$ the set of eigenvalues of the homogeneous Dirichlet problem for the Laplace operator Δ in the domain Ω, numbered in non-increasing order, taking into account multiplicity, and by $\{\varphi_k\}$ denote the set of orthonormal (in the sense of L^2) of the corresponding eigenvectors. Then

$$\ker L = \begin{cases} \{0\}, & \text{if } æ^{-1} \notin \{\lambda_k\}, \\ \operatorname{span}\{\varphi_l : æ^{-1} = \lambda_l\}. & \end{cases}$$

In case $\ker L = \{0\}$ there is an operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$, and so the operator $M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ $(L, 0)$-bounded. If $\ker L \neq \{0\}$ take the vector $\varphi \in \ker L \setminus \{0\}$, i.e.

$$\varphi = \sum_{æ^{-1} = \lambda_l} a_l \varphi_l, \quad a_l \in \mathbb{R}, \quad \sum_{æ^{-1} = \lambda_l} |a_l| > 0.$$

Since

$$M\varphi = \nu \sum_{æ^{-1} = \lambda_l} \lambda_l a_l \varphi_l \notin \operatorname{im} L,$$

then the vector φ has no M-adjoint vectors, and the assertion follows from theorem 3.

In what follows, we need regularity theorem [12]:

Lemma 3. *Let $f \in C^\infty(\mathbb{R})$ and $m > n/2$. Then $F \in C^\infty(W_2^m)$, where the operator $F : u \to f(u)$.*

Lemma 4. *Let the function $K \in C^\infty(\mathbb{R})$ and let $m+2 > n/2$. Then the operator $N : u \to K(u)$ belongs to the class $C^\infty(\mathfrak{U};\mathfrak{F})$.*

By virtue of lemma 3 the operator $N \in C^\infty(\mathfrak{U})$, and in view of the continuity of embedding $\mathfrak{U} \hookrightarrow \mathfrak{F}$, $N \in C^\infty(\mathfrak{U};\mathfrak{F})$.

So, reduction of problem (1) – (3) to (10), (11) is finished. By lemma 1 construct a projector

$$P = \begin{cases} \mathbb{I}, \quad \text{if} \quad æ^{-1} \notin \{\lambda_l\}, \\ \mathbb{I} - \sum\limits_{æ^{-1}=\lambda_l} \langle \cdot, \varphi_l \rangle \varphi_l, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is an inner product in L^2. The projector Q has the same form, but is defined on the space $\mathfrak{F}$. Fix $m > n/2 - 2$ and construct the set

$$\mathfrak{M} = \begin{cases} \mathfrak{U}, \quad \text{if} \quad æ^{-1} \notin \{\lambda_l\}, \\ \{u \in \mathfrak{U} : \langle Mu - N(u), \varphi_l \rangle = 0, \quad æ^{-1} = \lambda_l\} \end{cases}$$

and the space

$$\mathfrak{U}^1 = \begin{cases} \mathfrak{U}, \quad \text{if} \quad æ^{-1} \in \{\lambda_l\}, \\ \{u \in \mathfrak{U} : \langle Mu - N(u), \varphi_l \rangle = 0, \quad æ^{-1} = \lambda_l\}. \end{cases}$$

In the case of $æ^{-1} \notin \{\lambda_l\}$ the set $\mathfrak{M}$ is obviously a smooth Banach C^∞-manifold. In the case of $æ^{-1} \in \{\lambda_l\}$ this is yet to be proven.

Theorem 5. [33] *(i) For any $æ^{-1} \notin \{\lambda_k\}$, $\nu \in \mathbb{R} \setminus \{0\}$, $m > n/2 - 2$, $u_0 \in \mathfrak{U}$ and some $T \in \mathbb{R}_+$ there exists a unique solution $u \in C^\infty((-T,T);\mathfrak{U})$ of problem (1) – (3).*

(ii) Let for $æ \in \{\lambda_k\}$, $\nu \in \mathbb{R} \setminus \{0\}$, $m > n/2 - 2$ set $\mathfrak{M}$ at the point u_0 be a Banach C^∞-manifold. Then for some $T \in \mathbb{R}_+$ there exists a unique solution $u \in C^\infty((-T,T);\mathfrak{M})$ of problem (1) – (3).

Theorem 5 follows directly from theorem 4. We only note that in our case the operator $H \equiv \mathbb{O}$, and therefore, any solution to problem (1) – (3) necessarily turns out to be quasi-stationary trajectory.

4. Relatively Polynomially Bounded Operator Pencils

The statements presented in this paragraph were obtained in the works of A.A. Zamyshlyaeva [38, 45]. Let $\mathfrak{U}, \mathfrak{F}$ be Banach spaces and operators $A, B_0, B_1, \ldots, B_{n-1} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$. By $\vec{B}$ denote the pencil formed by operators $B_{n-1}, \ldots, B_1, B_0$. The sets $\rho^A(\overrightarrow{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and $\sigma^A(\overrightarrow{B}) = \overline{\mathbb{C}} \setminus \rho^A(\vec{B})$ are called an *A-resolvent set* and an *A-spectrum* of the pencil $\overrightarrow{B}$ respectively. The operator-function of a complex variable $R^A_\mu(\overrightarrow{B}) = (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1}$ with the domain $\rho^A(\overrightarrow{B})$ is called an *A-resolvent* of the pencil $\overrightarrow{B}$.

Definition 7. The operator pencil $\vec{B}$ is called *polynomially bounded with respect to an operator A* (*or polynomially A-bounded*) if

$$\exists a \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{F};\mathfrak{U})).$$

Remark 2. If there exists an operator $A^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$ then the pencil $\vec{B}$ is *A-bounded.*

Lemma 5. [38] *Let the operator pencil $\vec{B}$ be polynomially A-bounded and condition*

$$\int\limits_\gamma \mu^k R^A_\mu(\vec{B}) d\mu \equiv \mathbb{O}, \quad k = 0, 1, \dots, n-2, \tag{14}$$

where the circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$, be fulfilled. Then the operators

$$P = \frac{1}{2\pi i}\int\limits_\gamma R^A_\mu(\vec{B})\mu^{n-1} A d\mu, \quad Q = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-1} A R^A_\mu(\vec{B}) d\mu$$

are projectors in spaces $\mathfrak{U}$ and $\mathfrak{F}$ respectively.

Denote $\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{U}^1 = \operatorname{im} P$, $\mathfrak{F}^1 = \operatorname{im} Q$. According to lemma 5 $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. By A^k (B^k_l) denote restriction of operators A (B_l) onto $\mathfrak{U}^k$, $k = 0, 1;$ $l = 0, 1, \dots, n-1$.

Theorem 6. [38] *Let the operator pencil $\vec{B}$ be polynomially A-bounded and condition (14) be fulfilled. Then*
(i) $A^k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0, 1$, $l = 0, 1, \dots, n-1$;
(iii) operator $(A^1)^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$ exists;
(iv) operator $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$ exists.

Using theorem 6 construct operators $H_0 = (B^0_0)^{-1}A^0 \in \mathcal{L}(\mathfrak{U}^0)$, $H_1 = (B^0_0)^{-1}B^0_1 \in \mathcal{L}(\mathfrak{U}^0)$,..., $H_{n-1} = (B^0_0)^{-1}B^0_{n-1} \in \mathcal{L}(\mathfrak{U}^0)$ and $S_0 = (A^1)^{-1}B^1_0 \in \mathcal{L}(\mathfrak{U}^1)$, $S_1 = (A^1)^{-1}B^1_1 \in \mathcal{L}(\mathfrak{U}^1)$, ..., $S_{n-1} = (A^1)^{-1}B^1_{n-1} \in \mathcal{L}(\mathfrak{U}^1)$.

Definition 8. Define the *family of operators* $\{K^1_q, K^2_q, \dots, K^n_q\}$ as follows:

$$K^s_0 = \mathbb{O},\ s \neq n,\ K^n_0 = \mathbb{I},$$
$$K^1_1 = H_0,\ K^2_1 = -H_1, \dots, K^s_1 = -H_{s-1}, \dots, K^n_1 = H_{n-1},$$
$$K^1_q = K^n_{q-1}H_0,\ K^2_q = K^1_{q-1} - K^n_{q-1}H_1, \dots, K^s_q = K^{s-1}_{q-1} - K^n_{q-1}H_{s-1}, \dots,$$
$$K^s_q = K^{n-1}_{q-1} - K^n_{q-1}H_{n-1},\ q = 1, 2, \dots.$$

The A-resolvent can be represented by a Laurent series

$$(\mu^n A - \mu^{n-1}B_{n-1} - \dots - \mu B_1 - B_0)^{-1} = -\sum_{q=0}^{\infty} \mu^q K^n_q (B^0_0)^{-1}(\mathbb{I} - Q) +$$

$$+\sum_{q=1}^{\infty} \mu^{-q}(\mu^{n-1}S_{n-1} + \dots + \mu S_1 + S_0)^q {A^1}^{-1} Q.$$

Using this representation we classify the character of the infinity point of the A-resolvent of the operator pencil $\vec{B}$.

Definition 9. The point ∞ is called

- *a removable singular point of an A-resolvent of the pencil $\vec{B}$*, if $K_1^s \equiv \mathbb{O}, s = 1, 2, \ldots, n$;
- *a pole of order $p \in \mathbb{N}$ of an A-resolvent of the pencil $\vec{B}$*, if $\exists\, p$ such that $K_p^s \not\equiv \mathbb{O}, s = 1, 2, \ldots, n$, but $K_{p+1}^s \equiv \mathbb{O}, s = 1, 2, \ldots, n$;
- *an essential singular point of an A-resolvent of the pencil $\vec{B}$*, if $K_q^n \not\equiv \mathbb{O}$ for all $q \in \mathbb{N}$.

Further a removable singular point of an A-resolvent of the pencil $\vec{B}$ will be called a pole of order 0, for brevity. If the operator pencil $\vec{B}$ is polynomially A-bounded and the point ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of an A-resolvent of the pencil $\vec{B}$ then the operator pencil $\vec{B}$ is called *polynomially (A, p)-bounded.*

Theorem 7. [42] *Let A, B_{n-1}, $\ldots$, B_1, $B_0 \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$ and A be a Fredholm operator. Then the following statements are equivalent:*

(i) The lengths of all chains of the $\vec{B}$-adjoined vectors of the operator A are bounded by number $(p + n - 1) \in \{0\} \cup \mathbb{N}$ and the chain of length $(p + n - 1)$ exists.

(ii) The operator pencil $\vec{B}$ is polynomially (A, p)-bounded.

5. Semilinear Sobolev Type Equations of Higher Order

Consider the Cauchy problem

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, n-1, \tag{15}$$

for a semilinear Sobolev type equation of higher order

$$Au^{(n)} = B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0 u + N(u), \tag{16}$$

where operators $A, B_{n-1}, B_{n-2}, \ldots, B_0 \in \mathcal{L}(\mathfrak{U}; \mathfrak{F}), N \in C^\infty(\mathfrak{U}; \mathfrak{F})$, and $\mathfrak{U}, \mathfrak{F}$ are Banach spaces.

Definition 10. If a vector-function $u \in C^\infty((-\tau, \tau); \mathfrak{U})$, $\tau \in \mathbb{R}_+$ satisfies equation (16) then it is called *a solution of this equation.* If the vector-function satisfies in addition condition (15) then it is called *a solution of problem* (15), (16).

Definition 11. The set $\mathfrak{P}$ is called a *phase space of (16),* if
(i) for all $(u_0, u_1, \ldots, u_{n-1}) \in T^{n-1}\mathfrak{P}$ there exists a unique solution of (15), (16);
(ii) a solution $u = u(t)$ of (16) lies in $\mathfrak{P}$ as a trajectory, i.e. $u(t) \in \mathfrak{P}$ for all $t \in (-\tau, \tau)$.

If $\ker A = \{0\}$ then equation (16) can be reduced to an equivalent equation

$$u^{(n)} = F(u, \dot{u}, \ldots, u^{(n-1)}),$$

where $F(u, \dot{u}, \ldots, u^{(n-1)}) = A^{-1}(B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0 u + N(u))$ is a mapping of class C^∞ by construction. The existence of a unique solution u of (15), (16) for all $(u_0, u_1, \ldots, u_{n-1})$ follows from the classical Cauchy theorem.

Let $\ker A \neq \{0\}$ and operator pencil $\vec{B}$ be $(A,0)$-bounded, then by theorem 6 equation (16) can be reduced to an equivalent system of equations

$$\begin{cases} 0 = (\mathbb{I} - Q)(B_0 + N)(u^0 + u^1), \\ \dfrac{d^n}{dt^n} u^1 = A_1^{-1} Q \left(B_{n-1} \dfrac{d^{n-1}}{dt^{n-1}} + B_{n-2} \dfrac{d^{n-2}}{dt^{n-2}} + \ldots + B_0 + N\right)(u^0 + u^1), \end{cases} \tag{17}$$

where $u^1 = Pu, u^0 = (I - P)u$.

Now consider a set $\mathfrak{M} = \{u \in \mathfrak{U} : (I - Q)(B_0 u + N(u)) = 0\}$. Let the set $\mathfrak{M}$ be not empty, i.e. there is a point $u_0 \in \mathfrak{M}$. Denote ${u_0}^1 = Pu \in \mathfrak{U}^1$. The set $\mathfrak{M}$ is called *a Banach C^k-manifold at point* u_0 if there exist neighborhoods $\mathcal{O} \subset \mathfrak{M}$ and $\mathcal{O}^1 \subset \mathfrak{U}^1$ of points u_0 and u_0^1 respectively and a C^k-diffeomorphism $\delta : \mathcal{O}^1 \to \mathcal{O}$ such that δ^{-1} is a restriction of projector P on $\mathcal{O}$. The set $\mathfrak{M}$ is called *a Banach C^k-manifold* modelled by the space $\mathfrak{U}^1$ if it is a Banach C^k-manifold at any point.

Let the following condition be fulfilled

$$(\mathbb{I} - Q)(B_0 + N'_{u_0}) : \mathfrak{U}^0 \to \mathfrak{F}^0 \quad \text{is a toplinear isomorfism.} \tag{18}$$

According to the implicit function theorem there exist neighborhoods $\mathcal{O}^0 \subset \mathfrak{U}^0$ and $\mathcal{O}^1 \subset \mathfrak{U}^1$ of points $u_0^0 = (\mathbb{I} - P)u_0, u_0^1 = Pu_0$ respectively and the operator $B \in C^\infty(\mathcal{O}^1; \mathcal{O}^0)$ such that $u_0^0 = B(u_0^1)$. Lets construct an operator $\delta = \mathbb{I} + B : \mathcal{O}^1 \to \mathfrak{M},\ \delta(u_0^1) = u_0$. Then the operator δ^{-1} together with the set $\mathcal{O}^1$ makes a map of $\mathfrak{M}$ and is a restriction of P on $\delta[\mathcal{O}^1] = \mathcal{O} \subset \mathfrak{M}$. Thus, we proved

Lemma 6. [42] *The set $\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(B_0 u + N(u)) = 0\}$ under condition (18) is a C^∞-manifold at point u_0.*

Lets act with the Frechet derivative $\delta^{(n)}_{(u_0^1, u_1^1, \ldots, u_{n-1}^1)}$ of order n on the second equation of system (17). Since $\delta(u^1) = u$ and

$$\delta^{(n)}_{(u_0^1, u_1^1, \ldots, u_{n-1}^1)} u^{1(n)} = \frac{d^n}{dt^n} \left(\delta(u^1)\right)$$

we obtain equation $u^{(n)} = F(u, \dot{u}, \ldots, u^{(n-1)})$, where

$$F(u, \dot{u}, \ldots, u^{(n-1)}) = \delta^{(n)}_{(u_0^1, u_1^1, \ldots, u_{n-1}^1)} A^{-1} Q (B_{n-1} u^{(n-1)} + B_{n-2} u^{(n-2)} + \ldots$$
$$+ B_0 u + N(u)) \in C^\infty(\mathfrak{U}).$$

Therefore, we get

Theorem 8. [42] *Let the operator pencil $\vec{B}$ be $(A,0)$-bounded, $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$ and condition (18) be fulfilled. Then for any $(u_0, u_1, \ldots, u_{n-1}) \in T^{n-1}\mathfrak{M}$ there exists a unique solution of problem (15), (16) lying in $\mathfrak{M}$ as trajectory.*

6. Mathematical Model of Ion-Acoustic Waves in Plasma

As a model example, consider problem (4), (5). In order to reduce mathematical model (4), (5) to problem (15), (16) set

$$\mathfrak{U} = \{u \in W_2^{l+2}(\Omega) : u(x) = 0, x \in \partial\Omega\}, \quad \mathfrak{F} = W_2^l(\Omega).$$

Define operators $A = \Delta - \lambda$, $B_2 = (\lambda' - \Delta)$, $B_0 = -\alpha\frac{\partial^2}{\partial x_3^2}$, $B_3 = B_1 = \mathbb{O}$. Operators A, B_3, B_2, B_1, B_0 are $\in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ for all $l \in \{0\} \cup \mathbb{N}$. Denote the eigenvectors of the Dirichlet problem (5) for the Laplace operator by $\varphi_{kmn} = \left\{\sin\frac{\pi k x_1}{a}\sin\frac{\pi m x_2}{b}\sin\frac{\pi n x_3}{c}\right\}$, where k, m, $n \in \mathbb{N}$ and denote the eigenvalues by $\lambda_{kmn} = -\sqrt{\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2 + \left(\frac{\pi n}{c}\right)^2}$. The spectrum $\sigma(\Delta)$ is negative, discrete, finite and tends only to $-\infty$. Since $\{\varphi_{kmn}\} \subset C^\infty(\Omega)$ we obtain

$$\mu^4 A - \mu^3 B_3 - \mu^2 B_2 - \mu B_1 - B_0 =$$
$$= \sum_{k,m,n=1}^{\infty} [(\lambda_{kmn} - \lambda)\mu^4 + (\lambda_{kmn} - \lambda')\mu^2 - \alpha\left(\frac{\pi n}{c}\right)^2] < \varphi_{kmn}, \cdot > \varphi_{kmn},$$

where $< \cdot, \cdot >$ is an inner product in $L^2(\Omega)$.

Remark 3. In the case when (i) $\lambda \notin \sigma(\Delta)$ the A-spectrum of pencil $\vec{B}$ $\sigma^A(\vec{B}) = \{\mu^j_{rmn} : r, m, n \in \mathbb{N}, j = 1, ..., 4\}$, where μ^j_{rmn} are the roots of equation

$$(\lambda_{rmn} - \lambda)\mu^4 + (\lambda_{rmn} - \lambda')\mu^2 - \alpha\left(\frac{\pi n}{c}\right)^2 = 0. \tag{19}$$

In the case when (ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$ the A-spectrum of pencil $\vec{B}$ $\sigma^A(\vec{B}) = \{\mu^j_{l,k} : k \in \mathbb{N}\}$, where $\mu^j_{l,k}$ are the roots of equation (19) with $\lambda = \lambda_l$. In the case when (iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$ the A-spectrum of pencil $\vec{B}$ $\sigma^A(\vec{B}) = \{\mu^j_{l,k} : k \in \mathbb{N}, k \neq l\}$.

Check condition (14). In case (i) there exists $A^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$ therefore condition (14) is fulfilled.

In case (ii)

$$\frac{1}{2\pi i}\int_\gamma \sum_{k,m,n=1}^{\infty} \frac{\mu^r < \varphi_{kmn}, \cdot > \varphi_{kmn} d\mu}{(\lambda_{kmn} - \lambda)\mu^4 + (\lambda_{kmn} - \lambda')\mu^2 - \alpha(\frac{\pi n}{c})^2} =$$
$$= \frac{1}{2\pi i}\int_\gamma \sum_{k,m,n=1}^{\infty} \frac{\mu^r < \varphi_{kmn}, \cdot > \varphi_{kmn} d\mu}{(\lambda_{kmn} - \lambda')\mu^2 - \alpha(\frac{\pi n}{c})^2} \neq 0,$$

when $r = 1$, therefore condition (14) is not fulfilled and this case is excluded from further considerations. In case (iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$ condition (14) is fulfilled.

Lemma 7. [42] *Let (i)* $\lambda \notin \sigma(\Delta))$ *or (ii)* $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$. *Then the pencil* $\vec{B}$ *is polynomially* $(A, 0)$*-bounded.*

In case (i) $\ker A = \{0\}$ that is, the operator A has no eigenvectors and, by remark 2 the pencil $\vec{B}$ is $(A, 0)$-bounded.

In case (ii) $\lambda \in \sigma(\Delta)$ and $\lambda = \lambda'$ construct the chain of $\vec{B}$-adjoined vectors of an eigenvector $\varphi_0 = \sum\limits_{\lambda=\lambda_{kmn}} a_{kmn}\varphi_{kmn} \in \ker A \setminus \{0\}$. Since $B_3 = B_1 = \mathbb{O}$ the first three $\vec{B}$-adjoined vectors can be taken equal to zero. On the fourth we obtain

$$B_0\varphi_0 = B_0(\sum_{\lambda=\lambda_{kmn}} a_{kmn}\varphi_{kmn}) = -\alpha\left(\frac{\pi n}{c}\right)^2 \sum_{\lambda=\lambda_{kmn}} a_{kmn}\varphi_{kmn} \notin \mathrm{im}A,$$

since $\sum\limits_{\lambda=\lambda_{kmn}} |a_{kmn}| > 0$.

Therefore the eigenvector φ_0 doesn't have a $\vec{B}$-adjoined vector of order four, the length of the chains of $\vec{B}$-adjoined vectors of operator A is bounded by three, and the chain of length three exists.

Construct projectors. In case (i) $P = \mathbb{I}$ and $Q = \mathbb{I}$. In case (ii)

$$P = \mathbb{I} - \sum_{\lambda=\lambda_{kmn}} < \varphi_{kmn}, \cdot > \varphi_{kmn},$$

and the projector Q has the same form but it is defined on space $\mathfrak{F}$. Construct the set

$$\mathfrak{M} = \{u \in \mathfrak{U} : \sum_{\lambda=\lambda_{kmn}} < \alpha \left(\frac{\pi n}{c}\right)^2 u + \Delta(u^3), \varphi_{kmn} > \varphi_{kmn} = 0\}.$$

By theorem 8 we have

Theorem 9. [42] *(i) Let $\lambda \notin \sigma(\Delta)$, $(u_0, u_1, \ldots, u_{n-1}) \in \mathfrak{U}^n$. Then for some $\tau = \tau(u_0, u_1, \ldots, u_{n-1}) > 0$ there exists a unique solution $u \in C^n\big((-\tau, \tau), \mathfrak{U}\big)$ of problem (4), (5).*
(ii) Let $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$, $(u_0, u_1, \ldots, u_{n-1}) \in T^{n-1}\mathfrak{M}$ and condition (18) be fulfilled. Then for some $\tau = \tau(u_0, u_1, \ldots, u_{n-1}) > 0$ there exists a unique solution $u \in C^n\big((-\tau, \tau), \mathfrak{M}\big)$ of problem (4), (5).

7. Mathematical Model of Waves Propagation in Shallow Water

In some particular cases of a non-linear term in equation, one can not only answer the question of the existence and uniqueness of a solution, such as [41], but also find this solution. A detailed algorithm is described in [3], in this section we present only the main steps in finding a solution to problem (6) – (8).

For the solution, we need several function spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain with the boundary $\partial\Omega$ of class C^∞, denote $Q = \Omega \times (0, T)$. Define spaces $L^4(\Omega)$, $H_0^1(\Omega)$ and denote $B = L^4(\Omega) \cap H_0^1(\Omega)$, $D = H^1(\Omega) \cap \operatorname{coim} L$ (where $\operatorname{coim} L = H^1(\Omega) \ominus \ker L$).

The operator $\Delta : H^1(\Omega) \to H^{-1}(\Omega)$ is given by formula

$$\langle \Delta u, v\rangle = -\int_\Omega (\nabla u \nabla v) dx.$$

Introduce the notation of the operators

$$\langle Lu, v\rangle = \int_\Omega (\nabla u \nabla v + \lambda uv) dx, \;\; \langle Mu, v\rangle = \alpha^2 \int_\Omega (\nabla u \nabla v) dx, \;\; \langle N(u), v\rangle = \int_\Omega u^3 v dx.$$

In addition, define distribution spaces (functions with values in a Banach space) $L^\infty(0, T; B)$ and $L^\infty(0, T; L^2(\Omega))$. Construct dual spaces using the Dunford–Pettis theorem: $(L^\infty(0, T; B))^* \simeq L^1(0, T; L^{\frac{4}{3}})(\Omega) \cup H^{-1}(\Omega))$ and $(L^\infty(0, T; D))^* \simeq L^1(0, T; D^*)$.

Let λ_k be the eigenvalues of the homogeneous Dirichlet problem (7)) for the operator Δ, numbered in nonincreasing order with multiplicity taken into account, and φ_k be the corresponding eigenvectors. Moreover, the linear span $\operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ is dense in B for $m \to \infty$ and is orthonormal (in the sense of an inner product in $L^2(\Omega)$).

Theorem 10. [3] *Let* $\lambda \in [\lambda_1, +\infty)$, $(u_0, u_1) \in T\mathfrak{P}$, *where* $u_0 \in B = H_0^1(\Omega) \cap L^4(\Omega)$ *and* $u_1 \in L^2(\Omega) \cap \mathrm{coim} L$. *Then there exists a solution* $u = u(x,t)$ *of problem (6) – (8) such that* $u \in L^\infty(0,T; H_0^1(\Omega) \cap L^4(\Omega))$ *and* $\dot{u} \in L^\infty(0,T; L^2(\Omega) \cap \mathrm{coim} L)$.

Below we give a sketch of the proof. The solution of problem (6) – (8) will be sought in the form of the Galerkin approximation

$$u^m(t) = \sum_{k=1}^{m} a_k^m(t)\varphi_k. \tag{20}$$

Substituting the approximate solution (20) into equation (6) and multipliyng scalarly by the basis functions $\{\varphi_k\}_{k=1}^m$, we get

$$\langle L\ddot{u}^m, \varphi_k\rangle - \alpha^2\langle \Delta u^m, \varphi_k\rangle + \langle (u^m)^3, \varphi_k\rangle = 0, \quad 1 \le k \le m. \tag{21}$$

Using the series expansions of the initial functions in terms of basis functions, we obtain the initial conditions for the system of algebraic differential equations (21)

$$a_k^m(0) = \beta_k^m, \quad \dot{a}_k^m(0) = \gamma_k^m, \quad 1 \le k \le m, \tag{22}$$

where $u_0^m = \sum\limits_{k=1}^{m} \beta_k^m \varphi_k \to u_0$ in B for $m \to \infty$, and $u_1^m = \sum\limits_{k=1}^{m} \gamma_k^m \varphi_k \to u_1$ in $L^2(\Omega)$ as $m \to \infty$. The existence of a unique local solution $u^m = u^m(t,x)$, $t \in [0, t^m]$ was proved.

After that, a priori estimates were obtained as follows. Multiplying equation (21) by $\dot{a}_k^m(t)\,(1 \le k \le m)$ and summing over k from 1 to m, we get

$$\langle L\ddot{u}^m, \dot{u}^m\rangle - \alpha^2\langle \Delta u^m, \dot{u}^m\rangle + \langle (u^m)^3, \dot{u}^m\rangle = 0.$$

Introduce a norm in the space D ($L^2(\Omega) = \mathrm{coim}\, L \oplus \ker L$) $|\dot{u}|_{H^1}^2 = \langle L\dot{u}, \dot{u}\rangle$. By the Courant principle, this norm is equivalent to the norm induced by the space $H^1(\Omega)$. Using the self-adjointness of L, Δ and integrating it on the segment $[0,t], t \le t_m$ we obtain

$$|\dot{u}^m|^2 + \alpha^2\|u^m\|_{H_0^1}^2 + \frac{1}{2}\|u^m\|_{L^4}^4 \le C. \tag{23}$$

The constant C does not depend on t_m and hence $t_m = T$.

Remark 4. Due to inequality (23) for $m \to \infty$, the sequence of functions $\dot{u}_m$ is bounded in the space $L^\infty(0,T; L^2(\Omega))$ and u is bounded in $L^\infty(0,T;B)$.

Since the sequence $\{(u^{m_l})^3\}$ is bounded in the space $L^\infty(0,T; L^{4/3}(\Omega))$, we have

$$(u^{m_l})^3 \to z \text{ *-weakly in } L^\infty(0,T; L^{4/3}(\Omega)). \tag{24}$$

Moreover, it can be shown that $z = u^3$.

Now we can pass term by term to the limit in equality (21), setting $m_l = l$. Let k be fixed and $l > k$, we get

(i) due to the density of the system of functions $\{\varphi_k\}_{k=1}^m$ in the space B for $m \to \infty$, and the arbitrariness of the choice of φ_k, we have equality for arbitrary $v \in B$

$$\frac{d^2}{dt^2}\langle Lu, v\rangle + \alpha^2\langle \nabla u, \nabla v\rangle + \langle u^3, v\rangle = 0. \tag{25}$$

(ii) $u(0) = u_0$;
(iii) $\langle \dot{u}(0), \varphi_k \rangle = \langle u_1, \varphi_k \rangle, \quad \forall\, k.$

Remark 5. Under the conditions of the theorem 10 and the Rellich–Kondrashov theorem, the solution to problem (6) – (8) is unique.

Conclusion

Instead of the Cauchy condition in all mathematical models, one can consider the Showalter–Sidorov condition

$$P(u^{(k)}(0) - u_k) = 0, \quad k = 0, 1, \ldots, n-1, \tag{26}$$

where P is a projector along the kernel of the operator at the highest derivative with respect to t. Condition (26) is a natural generalization of the Cauchy condition for Sobolev type equations.

Further directions of development are seen in the study of semilinear Sobolev type equations with additive "white noise" [8, 9, 13], nonlinear inverse problems [43], as well as the study of multipoint initial-final problems [11, 23, 40].

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ПОЛУЛИНЕЙНЫЕ МАТЕМАТИЧЕСКИЕ МОДЕЛИ СОБОЛЕВСКОГО ТИПА

А.А. Замышляева[1]**, *Е.В. Бычков***[1]
[1]Южно-Уральский государственный университет, г. Челябинск, Российская Федерация

Статья содержит обзор результатов, полученных в научной школе Георгия Анатольевича Свиридюка, в области полулинейных математических моделей соболевского типа. В работе приведены результаты о разрешимости задачи Коши и Шоуолтера – Сидорова для полулинейных уравнений соболевского типа первого, второго и высокого порядков, а также примеры неклассических моделей математической физики, такие, как обобщенная модель нелинейной фильтрации Осколкова, распространения ионно-акустических волн в плазме, распространения волн на мелкой воде, которые исследуются путем редукции к одной из вышеперечисленных абстрактных задач. Методы исследования полулинейных уравнений соболевского типа базируется на теории относительно p-ограниченных операторов для уравнений первого порядка по переменной t и теории относительно полиномиально ограниченных пучков операторов для уравнений второго и высокого порядка по переменной t. В работе применяется метод фазового пространства, заключающийся в редукции сингулярного уравнения к регулярному, определенному на некотором подпространстве исходного пространства, для доказательства теорем существования и единственности и метод Галеркина для построения приближенного решения.

*Ключевые слова: уравнение Осколкова; модифицированное уравнение Буссинеска; уравнение ионно-звуковых волн в плазме; полулинейные уравнения соболевского типа; относительно p-ограниченные операторы; относительно полиномиально ограниченные пучки операторов; метод Галеркина; *-слабая сходимость.*

Алена Александровна Замышляева, доктор физико-математических наук, профессор, кафедра «Прикладная математика и программирование», Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), zamyshliaevaaa@susu.ru.

Евгений Викторович Бычков, кандидат физико-математических наук, кафедра «Уравнения математической физики», Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), bychkovev@susu.ru.

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