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ON INVERSE PROBLEMS WITH POINTWISE OVERDETERMINATION FOR MATHEMATICAL MODELS OF HEAT AND MASS TRANSFER

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This article is a survey devoted to inverse problems of recovering sources and coefficients (parameters of a medium) in mathematical models of heat and mass transfer. The main attention is paid to well-posedness questions of the inverse problems with pointwise overdetermination conditions. The questions of this type arise in the heat and mass transfer theory, in environmental and ecology problems, when describing diffusion and filtration processes, etc. As examples, we note the problems of determining the heat conductivity tensor or sources of pollution in a water basin or atmosphere. We describe three types of problems. The first of them is the problem of recovering point or distributed sources. We present conditions for existence and uniqueness of solutions to the problem, show non-uniqueness examples, and, in model situations, give estimates on the number of measurements that allow completely identify intensities of sources and their locations. The second problem is the problem of recovering the parameters of media, in particular, the heat conductivity. The third problem is the problem of recovering the boundary regimes, i. e. the flux through a surface or the heat transfer coefficient.

Keywords: heat and mass transfer; mathematical modelling; parabolic equation; uniqueness; inverse problem; point source.

Dedicated to anniversary of Professor A.L. Shestakov

Introduction

Consider inverse problems with pointwise ovedetermination of recovering coefficients or sources in mathematical models of heat and mass transfer. We describe theoretical results in the case of the second order parabolic systems

$$u_t + A(t,x,D)u = f = \sum_{i=1}^{r} f_i(t,x)q_i(t) + f_0,\ (t,x) \in Q = (0,T) \times G\ (G \subset \mathbb{R}^n). \tag{1}$$

The operator A is an elliptic operator with matrix coefficients of the dimension $h \times h$ representable as

$$A(t,x,D)u = -\sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_ix_j} + \sum_{i=1}^{n} a_i(t,x)u_{x_i} + a_0u,$$

where a_{ij}, a_i are $h\times h$ matrices. The following initial and boundary conditions are imposed:

$$u|_{t=0} = u_0,\quad Bu|_S = g(t,x),\ S = (0,T) \times \Gamma,\ \Gamma = \partial G, \tag{2}$$

where $Bu = u$ or $Bu = \sum_{i=1}^{n} \gamma_i(t,x)u_{x_i} + \gamma_0 u$ or $Bu = \frac{\partial u}{\partial N} + \sigma u = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_j}(t,x)\nu_i + \sigma(t,x)u(t,x)$, where $\vec{\nu} = (\nu_1,\ldots,\nu_n)$ is the outward unit normal to Γ. The unknowns in (1), (2) are a solution u and the function $q_i(t)$ ($i = 1,2,\ldots,s$) occurring either into the right-hand side of (1) or the operators A and B. The overdetermination conditions for recovering the functions $\{q_i\}_{i=1}^{s}$ are as follows:

$$u|_{x=b_i} = \psi_i(t),\quad i = 1,2,\ldots,s, \tag{3}$$

where $\{b_i\}$ is a set of points in G or on Γ.

These problems arise in mathematical modelling of heat and mass transfer processes, diffusion, filtration, and in many other fields (see [1–3]). The function f on the right-hand side in (1) is referred to as a source function. Generally, there are three types of inverse problems, one of them is the problem of recovering sources, the second problem is the problem of recovering coefficients of A and the third problem is the problem of recovering functions occurring in the boundary condition (for instance, heat flux of the heat transfer coefficient). First, we pay attention to the case of source problems. Of course, we should refer to the fundamental articles by A.I. Prilepko and his followers. In particular, the works [4, 5] establish the existence and uniqueness theorem for solutions to the problem of recovering the source $f(t,x)q(t)$ with the overdetermination condition $u(x_0,t) = \psi(t)$ (x_0 is a point in G) and $h = 1$. Similar results are obtained in [6] for the problem of recovering lower-order coefficient $p(t)$ in equation (1). The Hölder spaces are used as the basic spaces in these articles. The results were generalized in the book [7, § 6.6,§ 9.4], in which the existence theory for problems (1) – (3) was developed in an abstract form with the operator A replaced with $-L$, where L is a generator of an analytic semigroup. The main results are based on the assumptions that the domain of L is independent of time and the unknown coefficients occur into the lower part of the equation nonlinearly. Under certain conditions, the existence and uniqueness theorems were proven locally in time in the spaces of functions continuously differentiable with respect to time. There are many articles devoted to problems (1) – (3) in model situations, especially in the case of $n = 1$ (see, for instance, [8–11]). In these articles different collections of coefficients are recovered with the overdetermination conditions of form (3), in particular, including boundary points b_i. In this case the boundary condition and the overdetermination condition define the Cauchy data at a boundary point. In the case of $n = 1$, many results are presented in [8]. Problems (1) – (3) were considered in our articles [12–15], where conditions on the data were weakened in contrast to those in [7, § 9.4] and the solvability questions were treated in the Sobolev spaces. Note that in the article [15] the unknowns occur in the right-hand side of (1) nonlinearly. In this case the right-hand side is of the form $f(t,x,u,\nabla u,\vec{q}(t))$ (see also [4]). If the functions $\{q_i\}_{i=1}^{s}$ occur into the main part of the operator A then we arrive at the classical problem of recovering the thermal conductivity tensor [16, 17]. There are comparatively small number of theoretical results in this case. In particular, we can refer to [8, Sect.4.3], [9, 11, 18, 19], where the existence and uniqueness theorems in Hölder spaces are established in the case of the heat conductivity depending on time for $n = 1$. The above-mentioned articles [12–15] deal with the existence and uniqueness theorems in the multidimensional case for the integral and pointwise overdetermination conditions. Note that there are extremely large number of numerical results devoted to problem (1) – (3). We can refer to the articles [20–22], where the heat conductivity or capacity are restored with the use of additional boundary data (for instance, additional Dirichlet data on the boundary), and to [23–26], where the unknowns are recovered with the use of temperature measurements (3). Proceed with the point sources problems, which, in contrast to the distributed sources case, are always ill-posed. In this case in problem (1) – (3), $f = \sum_{i=1}^{r} \delta(x - x_i)q_i(t) + f_0$, the intensities $q_i(t)$ of point sources, their locations x_i and the number m are quantities to be determined. There is a small number of theoretical results devoted to the solving these inverse problems. The main results are connected with numerical methods of solving the problem and many of them are far from justified. The problem is ill-posed and examples when the problem is not solvable or has many solutions are easily constructed. Very often the methods rely on reducing the problem to an optimal control problem and minimization of the corresponding objective functional [2,27,30,38]. However, it is possible that the corresponding functionals can have many local minima. Some theoretical results devoted to problem (1) – (3) are available in [39–43]. The stationary case is treated in [42], where the Dirichlet data are complemented with the Neumann data and these data allow to solve the problem on recovering the number of sources, their locations, and intensities using test functions and a Prony-type algorithm. Similar results are obtained in the multidimansional case for the parabolic source problem and thereby the identifiability of point sources is proven in the case of Cauchy data on the boundary of a spatial domain (i. e., the Dirichlet data on the boundary in addition to the Neumann data are given). Model problem (1) – (3) ($G = \mathbb{R}^n$) is considered in [43], where the explicit representation of solutions to the direct problem (the Poisson formula) and an auxiliary variational problem are used to determine numerically the quantities $\sum_i q_i r_{ij}^l$ (here $q_i(t) = \text{const}$ for all i and $r_{ij} = |x_i - y_j|$, $l = 1, 2, \ldots$). The quantities found allow to determine the points $\{x_i\}$ and the intensities q_i (see Theorem 2 and the corresponding algorithm in [43]). In the one-dimensional case, the uniqueness theorem for solutions to problem (1) – (3) with $m = 1$ is stated in [39]. Similar results are presented also in [44]. To define a solution uniquely (intensities and source locations) in the one-dimensional case, we need the condition that the sources and the measurement points $\{b_i\}$ alternate, and this requires some unavailable information in a practical situation. The work [45] presents non-uniqueness examples in the problems of recovering point sources. Some numerical methods used in solving the problems of recovering point sources are described in [46–48].

Next, we describe two classical problems of recovering the heat flux and the heat transfer coefficients. The former problem is the problem of determination of the function g in (2). The measurement points $\{b_i\}$ can belong to the boundary or to be inside the domain. In the latter case the problem becomes ill-posed in the classical sense. Main articles are devoted to this problem but only few of them consider the theory of these problems. In the article [49] the uniqueness theorem was obtained in the case of $n = 1$. The same case is studied in [9], where the existence and uniqueness theorem is presented in the problem of simultaneous determination of the heat flux and the leading coefficient in the equation depending on time. However, the case of $n = 1$ is much simpler for the study. One of the first articles devoted to problem (1) – (3) in the multidimensional case is the article [50] (see also [51]), where, for $Mu = u_t - \Delta u$ and $g = \psi(x)\varphi(t)$ (the unknown function is $\varphi(t)$), the existence and uniqueness theorem was obtained in Hölder classes in the case of points $\{b_i\}$ that belong to the boundary. This article also contains the uniqueness theorem in the problem of recovering the heat transfer coefficient depending on time (the coefficient γ_0 or σ in the operator B). The proof is based on reducing the problem to an Abel integral equation using an asymptotic of the Green function. It is possible that this approach is not applicable in the case of general parabolic systems. One more approach to the study of problem (1) – (3) is based on reducing the problem to an integral Volterra equation of the second kind. It can be found in [52], the existence and uniqueness theorem here is obtained with the use of the fixed point theorem in the case of the points $\{b_i\}$ that belong to the boundary. The approach allows to develop a new numerical method for solving the problem. Next, we refer to the article [53], where some existence and uniqueness results are obtained in the ill-posed case when the points $\{b_i\}$ are inside the domain. A solution is sought in the Sobolev space but some data of problem must belong to some class of infinitely differentiable functions whose Laplace transform decays sufficiently quickly. There are a lot of articles devoted to numerical solving the problem. Necessary bibliography and some results can be found in [54–56].

At present, we can find many articles devoted to numerical methods for solving the problem of recovering the heat transfer coefficient. The points $\{b_i\}$ in (3) can be interior points of the domain G (see [57–60]) or belong to the boundary [49, 61, 62]. The stationary case is treated in [34]. In the article [49] a parabolic system is considered and the heat transfer coefficients are constants (the uniqueness theorem is established and the numerical method is described). The heat transfer coefficient depending on time is determined numerically in [57, 58, 60]. In [57, 63] the heat transfer coefficient depends on x and the overdetermination data agree with the Dirichlet data on a part of a cylinder, i.e., we have the Cauchy data on a part of the lateral boundary of the cylinder. The heat transfer coefficient depending on all variables is calculated in [64] with the use of the Cauchy data on the lateral boundary of the cylinder. In this article the main attention is paid to the existence and uniqueness theorems for the above described inverse problems.

1. Preliminaries

Let E be a Banach space. The symbol $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$) stands for the space of strongly measurable functions defined on G with values in E and a finite norm $\|\|u(x)\|_E\|_{L_p(G)}$. The notations of the Sobolev spaces $W_p^s(G;E)$ and $W_p^s(Q;E)$ are conventional (see [65]). If $E=\mathbb{R}$ or $E=\mathbb{R}^n$ then the latter space is denoted by $W_p^s(Q)$. The definitions of the Hölder spaces $C^{\alpha,\beta}(\overline{Q}), C^{\alpha,\beta}(\overline{S})$ can be found, for example, in [66]. By the norm of a vector, we mean the sum of the norms of its coordinates. For a given interval $J=(0,T)$, set $W_p^{s,r}(Q)=W_p^s(J;L_p(G))\cap L_p\big(J;W_p^r(G)\big)$ and $W_p^{s,r}(S)=W_p^s(J;L_p(\Gamma))\cap L_p\big(J;W_p^r(\Gamma)\big)$. Let $(u,v)=\int\limits_G u(x)v(x)\,dx$ and denote by $B_\delta(b)$ a ball of the radius δ centered at b. The symbol $\rho(X,Y)$ stands for the distance between the sets X and Y.

2. Distributed Sources

The results presented in this section can be found in [13–15]. Here, we replace conditions (3) with the more general conditions

$$<u(b_i,t),e_i>=\psi_i(t),\quad i=1,2,\ldots,s, \tag{4}$$

where the symbol $<\cdot,\cdot>$ stands for the inner product in $\mathbb{C}^h$, $\{e_i\}$ is a set of vectors of unit length, and we allow coinciding points and vectors among the points $\{b_i\}$ as well as the vectors $\{e_i\}$. In this section, the operator B coincides with $Bu=\sum_{i=1}^n\gamma_i(t,x)u_{x_i}+\gamma_0 u$.

The definition of the inclusion $\Gamma\in C^s$ can be found in [66, Chapter 1]. Here, we assume that $\Gamma\in C^2$ and the parameter $p>n+2$ is fixed. The parameter $\delta>0$ is said to be admissible if $\overline{B_\delta(b_i)}\subset G$ for the interior points $b_i\in G$, $\overline{B_\delta(b_i)}\cap\overline{B_\delta(b_j)}=\emptyset$ for $b_i\neq b_j$, $i,j=1,2,\ldots,s$. We take a sufficiently small parameter δ and assume that $\Gamma_\delta\in C^3$. More exactly, for every point $b_i\in\Gamma$, there exists a neighborhood U (the coordinate neighborhood) of this point and a coordinate system y (local coordinate system) obtained by rotation and translation of the origin from the initial one such that the y_n-axis is directed as the interior normal to Γ at b_i and the equation of the boundary $U\cap\Gamma$ is of the form $y_n=\omega(y')$, $\omega(0)=0$, $|y'|<\delta$, $y'=(y_1,\ldots,y_{n-1})$; moreover, we have $\omega\in C^3(\overline{B'_\delta(0)})$ ($B'_\delta(0)=\{y':|y'|<\delta\}$), $G\cap U=\{y:\ |y'|<\delta, 0<y_n-\omega(y')<\delta_1\}$, $(\mathbb{R}^n\setminus G)\cap U=\{y:|y'|<\delta,-\delta_1<y_n-\omega(y')<0\}$. For the given domain G, the numbers δ,δ_1 are fixed and without loss of generality we can assume that $\delta_1>(M+1)\delta$, where M is the Lipschitz constant of the function ω. Assume that $Q^\tau=(0,\tau)\times G$, $G_\delta=\cup_i(B_\delta(x_i)\cap G)$, $Q_\delta=(0,T)\times G_\delta$, $Q_\delta^\tau=(0,\tau)\times G_\delta$, $S_\delta=(0,T)\times\Gamma_\delta$ $\Gamma_\delta=\cup_i B_\delta(x_i)\cap\Gamma$. Next, we fix an admissible parameter δ. Our conditions for the data are as follows.

$$a_{ij}\in C(\overline{Q}),\ a_k\in L_p(Q),\ \gamma_k\in C^{1/2,1}(\overline{S}),\ a_{ij}\in L_\infty(0,T;W^1_\infty(G_\delta)); \tag{5}$$

$$a_k\in L_p(0,T;W_p^1(G_\delta)),\ i,j=1,2,\ldots,n,\ k=0,1,\ldots,n. \tag{6}$$

The operator L is parabolic and the Lopatiskii condition holds. State these conditions. Introduce the matrix $A_0(t,x,\xi)=-\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j$ ($\xi\in\mathbb{R}^n$), and assume that there exists a constant $\delta_1 > 0$ such that the roots p of the polynomial $\det\big(A_0(t,x,i\xi)+pE\big)=0$ (E is the identity matrix) satisfy the condition

$$\mathrm{Re}\, p \leq -\delta_1|\xi|^2 \quad \forall\xi\in\mathbb{R}^n \quad \forall(x,t)\in Q. \tag{7}$$

The Lopatinskii condition can be stated as follows: for every point $(t_0,x_0)\in S$ and the operators $A_0(x,t,D)$ and $B_0(x,t,D)=\sum_{i=1}^n \gamma_i(t,x)\partial_{x_i}$ written in the local coordinate system y at this point, the system

$$\big(\lambda E + A_0(x_0,t_0,i\xi',\partial_{y_n})\big)v(z)=0, \quad B_0(x_0,t_0,i\xi',\partial_{y_n})v(0)=h_j, \tag{8}$$

where $\xi'=(\xi_1,\dots,\xi_{n-1})$, $y_n\in\mathbb{R}^+$, has a unique solution $C\big(\overline{\mathbb{R}}^+\big)$ decreasing at infinity for all $\xi'\in\mathbb{R}^{n-1}$, $|\arg\lambda|\leq\pi/2$, and $h_j\in\mathbb{C}$ such that $|\xi'|+|\lambda|\neq 0$.

We also assume that there exists a constant $\varepsilon_1>0$ such that

$$Re\,(-A_0(t,x,\xi)\eta,\eta)\geq\varepsilon_1|\xi|^2|\eta|^2 \quad \forall\xi\in\mathbb{R}^n,\ \eta\in\mathbb{C}^h, \tag{9}$$

where the brackets $(\cdot,\cdot)$ denote the inner product in $\mathbb{C}^h$ (see [66, Definition 7, Section 8, Chapter 7]).

Let

$$|\det(\sum_{i=1}^n \gamma_i\nu_i)|\geq\varepsilon_0>0, \tag{10}$$

where ν is the outward unit normal to Γ, ε_0 is a positive constant, and

$$u_0(x)\in W_p^{2-2/p}(G),\ g\in W_p^{s_0,2s_0}(S),\ B(x,0)u_0(x)|_\Gamma=g(x,0)\ \forall x\in\Gamma, \tag{11}$$

where $s_0=1/2-1/2p$. Fix an admissible $\delta>0$. Construct functions $\varphi_i(x)\in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_i(x)=1$ in $B_{\delta/2}(x_i)$ and $\varphi_i(x)=0$ in $\mathbb{R}^n\setminus B_{3\delta/4}(x_i)$ and denote $\varphi(x)=\sum_{i=1}^r\varphi_i(x)$. Additionally it is assumed that

$$\varphi(x)u_0(x)\in W_p^{3-2/p}(G),\ \varphi g\in W_p^{s_1,2s_1}(S)\ (s_1=1-1/2p), \tag{12}$$

$$\Gamma\in C^2,\ \gamma_k\in C^{1,2}(\overline{S_\delta})\ (k=0,1,2,\dots,n). \tag{13}$$

Consider problem (1)-(3), where

$$A=L_0+\sum_{k=r+1}^s q_k(t)L_k,\ L_ku=-\sum_{i,j=1}^n a_{ij}^k(t,x)u_{x_jx_j}+\sum_{i=1}^n a_i^k(t,x)u_{x_i}+a_0^k(t,x)u,$$

and $k=0,r+1,r+2,\dots,s$. The unknowns q_i are sought in the class $C([0,T])$. Construct a matrix $B(t)$ of the order $s\times s$ with the rows

$$<f_1(t,b_j),e_j>,\dots,<f_r(t,b_j),e_j>,<-L_{r+1}u_0(t,b_j),e_j>,\dots,<-L_su_0(t,b_j),e_j>.$$

We suppose that

$$\psi_j\in C^1([0,T]),\quad <u_0(b_j),e_j>=\psi_j(0)\ (j=1,2,\dots,s),\ \gamma_l\in C^{1/2,1}(\overline{S})\cap C^{1,2}(\overline{S_\delta}), \tag{14}$$

$$a_{ij}^k\in C(\overline{Q})\cap L_\infty(0,T;W_\infty^1(G_\delta)),\ a_l^k\in L_p(Q)\cap L_\infty(0,T;W_p^1(G_\delta))\ (i,j=1,\dots,n), \tag{15}$$

$$f_i\in L_p(Q)\cap L_\infty(0,T;W_p^1(G_\delta))\ (i=0,1,\dots,r), \tag{16}$$

for some admissible $\delta > 0$, $p > n+2$, and $k = 0, r+1, \ldots, s, \quad l = 0, 1, \ldots n$;

$$a_i^k(t, b_l), f_i(t, b_l) \in C([0,T]) \tag{17}$$

for all possible values of i, k, l. We also need the following condition: there exists a number $\delta_0 > 0$ such that

$$|\det B(t)| \geq \delta_0 \text{ a.e. on } (0,T). \tag{18}$$

Note that the entries of the matrix B belong to the class $C([0,T])$. Consider the system

$$\psi_{jt}(0) + < L_0 u_0(0, x_j), e_j > - < f_0(0, x_j), e_j >=$$
$$= \sum_{k=1}^{m} q_{0k} < f_k(0, x_j), e_j > - \sum_{k=m+1}^{m_1} q_{0k} < L_k u_0(0, x_j), e_j >, \ j = 1, \ldots, s, \tag{19}$$

where the vector $\vec{q}_0 = (q_{01}, q_{02}, \ldots, q_{0s})$ is unknown. Under condition (18), this system is uniquely solvable. Let $A_1 = L_0 + \sum_{k=m+1}^{r} q_{0k} L_k$. The following theorem is valid.

Theorem 1. [14] *Let conditions* (9) – (18) *be satisfied. Moreover, assume that conditions* (7), (8) *hold for the operator* $\partial_t + A_1$. *Then there exists a number* $\tau_0 \in (0,T]$ *such that, on the interval* $(0, \tau_0)$, *there exists a unique solution* $(u, q_1, q_2, ..., q_s)$ *to problem* (1) – (3) *such that* $u \in L_p(0, \tau_0; W_p^2(G))$, $u_t \in L_p(Q^{\tau_0})$, $q_i(t) \in C([0, \tau^0])$, $i = 1, \ldots, s$. *Moreover,* $\varphi u \in L_p(0, \tau_0; W_p^3(G_\delta))$, $\varphi u_t \in L_p(0, \tau_0; W_p^1(G_\delta))$.

In the case of the unknown lower-order coefficients, the results can be reformulated in a more convenient form. In this case the operator A is assumed to be representable in the form

$$A = L_0 - \sum_{i=m+1}^{r} q_i(t) l_i, \ L_0 u = -\sum_{i,j=1}^{n} a_{ij}(t,x) u_{x_j x_j} + \sum_{i=1}^{n} a_i(t,x) u_{x_i} + a_0(t,x) u,$$
$$l_i u = \sum_{j=1}^{n} b_{ij}(t,x) u_{x_j} + b_{i0}(t,x) u. \tag{20}$$

Moreover, the rows of the matrix $B(t)$ of the order $r \times r$ are as follows:

$$< f_1(t, b_i), e_i >, \ldots, < f_m(t, b_i), e_i >, < l_{m+1} u_0(t, b_i), e_i >, \ldots, < l_r u_0(t, b_i), e_i > .$$

We suppose that

$$\psi_j \in W_p^1(0,T), \quad < u_0(b_j), e_j >= \psi_j(0), \quad j = 1, 2, \ldots, s, \tag{21}$$

$$f_i, b_{kj} \in L_\infty(0,T; W_p^1(G_\delta)) \cap L_\infty(0,T; L_p(G)), \quad f_0 \in L_p(Q) \cap L_p(0,T; W_p^1(G_\delta)), \tag{22}$$

for some admissible $\delta > 0$, where $i = 1, \ldots, r$, $j = 0, 1, \ldots, n$, $k = r+1, \ldots, s$. The remaining coefficients satisfy the conditions

$$a_{ij} \in C(\overline{Q}), \ a_k \in L_p(Q), \ \gamma_k \in C^{1/2,1}(\overline{S}) \cap C^{1,2}(\overline{S_\delta}), \ a_{ij} \in L_\infty(0,T; W_\infty^1(G_\delta)); \tag{23}$$

$$a_k \in L_p(Q) \cap L_p(0,T; W_p^1(G_\delta)), \ i,j = 1, 2, \ldots, n, \ k = 0, 1, \ldots, n. \tag{24}$$

The corresponding theorem is stated in the following form.

Theorem 2. [14] *Assume that the parabolicity condition and Lopatinskii condition* (7), (8) *for the operator* $\partial_t + L_0$, *conditions* (9) – (13), (18), (21) – (24) *for some admissible* $\delta > 0$ *hold and* $p > n+2$. *Then, for some* $\gamma_0 \in (0, T]$, *on the interval* $(0, \gamma_0)$, *there exists a unique solution* $(u, q_1, q_2, ..., q_s)$ *to problem* (1) – (3) *such that* $u \in L_p(0, \gamma_0; W_p^2(G))$, $u_t \in L_p(Q^{\gamma_0})$, $\varphi u \in L_p(0, \gamma_0; W_p^3(G))$, $\varphi u_t \in L_p(0, \gamma_0; W_p^1(G))$, $q_i(t) \in L_p(0, \gamma_0)$, $i = 1, \ldots, s$.

Remark 1. A slightly sharper results can be found in [12], where the operator B can be arbitrary, in particular, the boundary condition can coincide with the Dirichlet boundary condition, but the points $\{b_j\}$ are interior points of G. The quasilinear case is considered in [15].

3. Point Sources

In this section we present some results concerning with recovering of point sources. We consider the simple parabolic equation

$$u_t + Lu = \sum_{i=1}^{r} q_i(t)\delta(x - x_i) + f_0(t, x), \quad Lu = -\Delta u + \sum_{i=1}^{n} a_i(x)u_{x_i} + a_0(x)u, \tag{25}$$

where $(x, t) \in Q = (0, T) \times G$, G is a domain in $\mathbb{R}^n$ $(n = 1, 2, 3)$ with a boundary $\Gamma \in C^2$. We consider three cases $G = \mathbb{R}^n$, or $G = \mathbb{R}^n_+ = \{x : x_n > 0\}$, or G is a domain with a compact boundary. The unknowns are the functions $q_i(t)$. Equation (25) is completed with the initial and boundary conditions

$$Bu|_S = g, \quad u|_{t=0} = u_0(x), \quad S = (0, T) \times \Gamma, \tag{26}$$

where either $Bu = \frac{\partial u}{\partial \nu} + \sigma u$, or $Bu = u$ (ν is the outward unit normal to Γ), and the overdetermination conditions are as follows:

$$u(b_j, t) = \psi_j(t), \ j = 1, 2, \ldots, s. \tag{27}$$

The coefficients in (25) are real-valued.

First, we describe our conditions on the data and present the simplest existence theorem. Let $\vec{a} = (a_1, a_2)$ for $n = 2$ and $\vec{a} = (a_1, a_2, a_3)$ for $n = 3$. The brackets $(\cdot, \cdot)$ denote the inner product in $\mathbb{R}^n$. The coefficients in (25) are assumed to be real-valued and

$$a_i \in W^2_\infty(G) \ (i = 1, \ldots, n), \ \nabla\psi, \Delta\psi, a_0 \in L_\infty(G), \ \sigma \in C^1(\Gamma). \tag{28}$$

Fix a parameter $\lambda \in \mathbb{R}$ and assume that

$$e^{-\lambda t} g \in W_2^{1/4,1/2}(S), \ \text{if } Bu = \frac{\partial u}{\partial \nu} + \sigma u \ (\sigma \in C^1(\Gamma)), \tag{29}$$

$$e^{-\lambda t} g \in W_2^{3/4,3/2}(S), \ \text{if } Bu = u, \quad f_0 e^{-\lambda t} \in L_2(G), \tag{30}$$

$$u_0(x) \in W_2^1(G), \quad u_0(x)|_\Gamma = g(x, 0) \text{ if } Bu = u. \tag{31}$$

If the parameter λ is sufficiently large then under the above conditions there exists a unique solution $w_0 \in W_2^{1,2}(Q)$ to the problem

$$u_t + Lu = f_0(t, x), \quad Bu|_S = g, \quad u|_{t=0} = u_0(x) \tag{32}$$

such that $e^{-\lambda t}w_0 \in W_2^{1,2}(Q)$. Consider problem (25) – (27). After the change of variables $w = u - w_0$, we arrive at the simpler problem

$$w_t + Lw = \sum_{i=1}^{m} q_i(t)\delta(x - x_i), \tag{33}$$

$$Bw|_S = 0, \quad w|_{t=0} = 0, \tag{34}$$

$$w(b_j, t) = \psi_j(t) - w_0(t, b_j) = \tilde{\psi}_j(t), \ j = 1, 2, \ldots, s. \tag{35}$$

Assume that the functions $\tilde{\psi}_j(t)$ admit the representation

$$\tilde{\psi}_j(t) = \int_0^t V_{\delta_j}(t-\tau)\psi_{0j}(\tau)d\tau, \ \psi_{0j}e^{-\lambda t} \in L_2(0,T), \tag{36}$$

where $V_\gamma(t) = \frac{e^{-\gamma^2/4t}}{4\pi t}$ for $n = 2$ and $V_\gamma = \frac{\gamma e^{-\gamma^2/4t}}{2\sqrt{\pi}t^{3/2}}$ for $n = 3$.

Assume that $K = \{y \in G : \ \rho(y, \cup_{i=1}^m x_i) \leq \rho(y, \Gamma)\}$ for $G \neq \mathbb{R}^n$ and K is an arbitrary compact set otherwise. We also assume that all coefficients in (25) admit extensions on the whole $\mathbb{R}^n$ such that conditions (28) hold for the case of $G = \mathbb{R}^n$. If G is a domain with a compact boundary, then such an extension always exists. Denote

$$\varphi_j(x) = \frac{-1}{2}\int_0^1 (\vec{a}(b_j + \tau(x - b_j)), (x - b_j))\, d\tau.$$

Let $\delta_j = \min_i r_{ij}, j = 1, 2, \ldots, s$, where $r_{ij} = |x_i - b_j|$. Introduce the matrix A_0 with the entries $a_{ji} = e^{\varphi_j(x_i)}$ if $|x_i - b_j| = \delta_j$ and $a_{ji} = 0$ otherwise. We need the condition

$$\det A_0 \neq 0. \tag{37}$$

Fix $p \in (1, n/(n-1))$. Suppose that the space $W^1_{p,B}(G)$ agrees with $W^1_p(G)$ in the case of the Robin boundary conditions and with the subspace of $W^1_p(G)$ comprising the functions vanishing on Γ otherwise. $W^{-1}_{p,B}(G)$ is the dual space to $W^1_{q,B}(G)$ $(1/p + 1/q = 1)$.

Theorem 3. [68] *Assume that $T = \infty$, $r = s$, conditions* (28), (31), (37) *hold and $b_i \in K$ for $i = 1, 2, \ldots, s$. Then there exists $\lambda_0 \geq 0$ such that if $\lambda \geq \lambda_0$ and conditions* (29), (30), (36) *are fulfilled then there exists a unique solution to problem* (25) – (27) *such that $u = w_0 + w$, w_0 is a solution to problem* (32), *$e^{-\lambda t}w_0 \in W_2^{1,2}(Q)$, $e^{-\lambda t}\vec{q} \in L_2(0,\infty)$, $e^{-\lambda t}w \in L_2(0,\infty; W^1_{p,B}(G))$, $e^{-\lambda t}w_t \in L_2(0,\infty; W^{-1}_{p,B}(G))$, $e^{-\lambda t}w \in W_2^{1,2}(Q_\varepsilon)$ for all $\varepsilon > 0$.*

We note that the following condition is actually a necessary condition for the uniqueness of a solution to problem (25) – (27). If the condition is not satisfied then any number of the points $\{b_i\}$ does not ensure uniqueness of solutions (see examples below).

Condition (A). For $n = 2$, any three points $\{b_i\}$ do not belong to the same straight line and, for $n = 3$, any four points $\{b_i\}$ do not belong to the same plane.

Next, we describe some model situation in which $Lu = -\Delta u + \lambda_0 u$, $\lambda_0 \geq 0$, $G = \mathbb{R}^n$ and the functions q_i on the right-hand side of (25) are real constants.

Theorem 4. [45] *Let u_1, u_2 be two solutions to problem* (25) – (27) *from the class described in Theorem* 3 *with the right-hand sides in* (25) *of the form $\sum_{i=1}^{r_j} q_i^j \delta(x - x_i)$ ($q_i^j = const, j = 1, 2$). Assume that the condition* (A) *is satisfied and $s \geq 2r + 1$ in the case of $n = 2$ and $s \geq 3r + 1$ in the case of $n = 3$, where $r \geq \max(r_1, r_2)$ (i.e., there exists the upper bound* *for the number* $\max(r_1, r_2)$). *Then* $u_1 = u_2$, $r_1 = r_2$, *and* $q_i^1 = q_i^2$ *for all* i *that is a solution to the problem of recovering the number* m, *the points* x_i, *and the constants* q_i *is unique.*

We now present the corresponding examples showing the accuracy of the results obtained. The following example shows that if Condition (A) is not satisfied then the problem of recovering the intensities of sources (sinks) located at x_1, x_2 has a nonunique solution. At the same time, it is an example of the nonuniqueness in the problem of recovering the intensity of one source and its location. Note that the problem of determining the location of one source x_0 and its intensity $q(t)$ is simple enough and in order to uniquely recover these parameters we need two measurements in the case of $n = 1$ [44], three measurements in the case of $n = 2$ [67] and four measurements (that is $s = 4$ in (27)) in the case of $n = 4$ [69]. The smaller number of points does not allow to determine the parameters $q(t), x_0$ uniquely. We should also require that the point x_0 is situated between two measurement points in the case of $n = 1$ and Condition (A) holds in the case of $n = 2, 3$. The numerical solution of the problem of recovering one source is treated in the articles [28, 30–37, 41, 67].

Example 1. First we take $n = 3$, $G = \mathbb{R}^n$, $Lu = -\Delta u$. Let u be a solution to equation (25) satisfying the homogeneous initial value conditions with the right-hand side in (25) of the form

$$q(t)(\delta(x - x_1) - \delta(x - x_2)).$$

The Laplace transform of this solution to problem (25), (26) is written as

$$\hat{u} = \hat{q}(\lambda)\left(\frac{1}{4\pi|x - x_1|}e^{-\sqrt{\lambda}|x-x_1|} - \frac{1}{4\pi|x - x_2|}e^{-\sqrt{\lambda}|x-x_2|}\right).$$

Let P be the plane perpendicular to the segment $[x_1, x_2]$ and passing through its center. We have

$$\hat{u}(y, \lambda) \equiv 0 \ \ \forall y \in P.$$

Therefore, $u(y, t) = 0$ for all $y \in P$. Precisely the same example can be constructed in the case of $n = 2$. We take the perpendicular to the segment $[x_1, x_2]$ passing through its center rather than the plane P. Thus, if Condition (A) is not satisfied then any number of measurement points does not allow to determine the intensity and the location of the sources.

Example 2. Consider the case of $G = \mathbb{R}^n$, $Lu = -\Delta u$. In this case condition (27) with $s = 4$ in the case of $n = 2$ and $s = 6$ in the case of $n = 3$ does not allow to determine location of two sources and their intensities even if Condition (A) holds. We describe the example in the case of $n = 3$. The case of $n = 2$ is quite similar. Let u_1, u_2 be solutions to equation (25) satisfying the homogeneous initial value conditions in which the right-hand sides are of the form

$$q(t)\delta(x - x_1) + q(t)\delta(x - x_2), \ q(t)\delta(x - x_1^*) + q(t)\delta(x - x_2^*).$$

Let, for example, $n = 3$. Then the Laplace transforms of $\hat{u}_1, \hat{u}_2$ are as follows:

$$\hat{u}_1(x, \lambda) = \sum_{i=1}^{2} \frac{\hat{q}}{4\pi|x - x_i|}e^{-\sqrt{\lambda}|x-x_i|}, \ \ \hat{u}_2(x, \lambda) = \sum_{i=1}^{2} \frac{\hat{q}}{4\pi|x - x_i^*|}e^{-\sqrt{\lambda}|x-x_i^*|}. \qquad (38)$$

Here we use explicit representations of the fundamental solution to the Helmgoltz equation. We take $x_1 = (a, a, 0)$, $x_1^* = (a, -a, 0)$, $x_2 = (-a, -a, 0)$, $x_2^* = (-a, a, 0)$ $(a > 0)$. It is easy to see that the functions $\hat{u}_1, \hat{u}_2$ coincide at the points

$$y_1 = (M, 0, 0), y_2 = (-M, 0, 0), y_3 = (0, M, 0), y_4 = (0, -M, 0), y_5 = (0, 0, M), y_6 = (0, 0, -M),$$

where $M > 0$ and, thus, the problem of recovering the locations of 2 sources and their intensities admits several solutions in the case of $s = 6$. It follows from Theorem 4 that in the case of $s = 7$ the points x_1, x_2 and the intensities are determined uniquely (if Condition (A) holds and the intensities are constants).

4. Recovering of Heat Flux

Consider the parabolic equation

$$Mu = u_t + Lu = f(t,x), \quad (t,x) \in Q = (0,T) \times G, \ T \leq \infty, \tag{39}$$

where $Lu = -\Delta u + \sum_{i=1}^{n} a_i(x) u_{x_i} + a_0(x)u$, $G = \mathbb{R}^n_+$ or G is a domain with a compact boundary of the class C^2, and $n = 2, 3$. Equation (39) is completed with the initial-boundary conditions

$$Bu|_S = g(t,x) \ \ (S = (0,T) \times \Gamma), \quad u|_{t=0} = u_0(x), \tag{40}$$

where $Bu = \frac{\partial u}{\partial \nu} + \sigma(x)u$, and ν is the outward unit normal to Γ under the overdetermination conditions

$$u(t, b_i) = \psi_i(t) \ (i = 1, 2, \ldots, r), \tag{41}$$

where $\{b_i\}_{i=1}^r$ is a set of points that belong to G. Assume that $g(t,x) = \sum\limits_{j=1}^{r} \alpha_i(t)\Phi_i(x)$ for some known functions Φ_j, the problem consists in recovering both a solution to (39) satisfying (40) and (41) and the functions α_j, $j = 1, 2, \ldots, r$, characterizing g. Note that any function can be approximated by the sums of this form for a suitable choice of basis functions Φ_i.

There is a limited number of theoretical results devoted to problem (39) – (41). The problem is ill-posed in the Hadamard sense (see [70]) if the points $\{b_i\}_{i=1}^r$ are interior points of G. Let

$$\varphi_j(x) = \frac{1}{2} \int\limits_0^1 (\vec{a}(b_j + \tau(x - b_j)), (x - b_j)) \, d\tau. \tag{42}$$

For a given set of points $b_j \in G$ ($j = 1, 2, \ldots, r$), construct the point $b \in \Gamma$ such that $\delta_j = \rho(b_j, \Gamma) = |b - b_j|$.

Below, we assume that, for every $j = 1, 2, \ldots, r$, the set K_j consists of finitely many points and

$$\forall j = 1, 2, \ldots, r, \ \forall b \in K_j, \quad |\kappa_i|\delta_j < 1 \ (i = 1, 2) \text{ for } n = 3, \ |\kappa|\delta_j < 1 \text{ for } n = 2, \tag{43}$$

where κ_i are the principal curvatures of Γ at b for $n = 3$ and, respectively, κ is the curvature of Γ for $n = 2$ at b.

Let Ψ be the matrix with the entries $\Psi_{ji} = \sum\limits_{b \in K_j} \frac{\Phi_i(b) e^{-\varphi_j(b)}}{I_j(b)}$ $(i, j = 1, 2, \ldots, r)$. We assume that

$$\det \Psi \neq 0, \ \Phi_i(x) \in W_2^{1/2}(\Gamma), \tag{44}$$

$$\Phi_i(x) \in W_2^1(X_b) \text{ for } \ n = 2, \ \Phi_i(x) \in W_2^2(X_b) \text{ for } \ n = 3, \ b \in \cup_{j=1}^r K_j, \tag{45}$$

where $X_b = \overline{Y_b \cap \Gamma}$ (here Y_b is the coordinate neighborhood of b). We require also that

$$a_0 \in W_\infty^1(\cup_{b \in K_j}(Y_b \cap G)), \ \Gamma_\delta \in C^3, \ \sigma \in C^{3/2+\varepsilon}(\Gamma_\delta) \ (\varepsilon > 0), \tag{46}$$

$$u_0(x) \in W_2^1(G),\ e^{-\gamma_0 t} f \in L_2(Q). \tag{47}$$

Here $\Gamma_\delta = (\cup_{j=1}^r \cup_{b\in K_j} Y_b) \cap \Gamma$. The classical theory says that if condition (47) holds for some sufficiently large γ_0 then there exists a unique solution w_0 to problem (39) and (40), where $g = 0$, such that $e^{-\gamma_0 t} w_0 \in W_2^{1,2}(Q)$. Consider problem (39) – (41). Changing the variables $w = u - w_0$, we obtain the simpler problem

$$w_t + Lw = 0,\ Bw|_S = g(t,x),\quad w|_{t=0} = 0, \tag{48}$$

$$w(b_j,t) = \psi_j(t) - w_0(t,b_j) = \tilde{\psi}_j(t),\ j = 1,2,\ldots,r. \tag{49}$$

We assume that $\tilde{\psi}_j(t) \in L_2(0,T)$ and

$$\tilde{\psi}_j(t) = \int_0^t V_{\delta_j}(t-\tau)\psi_{0j}(\tau)d\tau,\ \psi_{0j}e^{-\gamma_0 t} \in \tilde{W}_2^{n/4}(0,T)\ (n = 2,3), \tag{50}$$

where $V_\gamma(t) = \frac{e^{-\gamma^2/4t}}{4\pi t}$ for $n = 2$ and $V_\gamma = \frac{\gamma e^{-\gamma^2/4t}}{2\sqrt{\pi}t^{3/2}}$ for $n = 3$. For $T = \infty$, condition (50) can be rewritten as

$$\sup_{\sigma>\gamma_0} \int_{-\infty}^{\infty} |\sigma + is|^{n/2} e^{Re\sqrt{p}\delta_j} |\mathcal{L}(\tilde{\psi}_j)(\sigma + is)|^2\, ds < \infty,\ \text{where } p = \sigma + is. \tag{51}$$

Theorem 5. [53] *Assume that $T = \infty$ and conditions (28), (43) – (45), and (46) for $n = 3$ hold. Then there exists $\lambda_0 \geq 0$ such that, if $Re\,\lambda = \gamma_0 \geq \lambda_0$ and conditions (47), (50) are fulfilled, then there exists a unique solution to problem (39) – (41) such that $e^{-\gamma_0 t}u \in W_2^{1,2}(Q)$, $e^{-\gamma_0 t}\alpha_i(t) \in W_2^{1/4}(0,T)$ $(i = 1,2,\ldots,r)$.*

Note that, for $T < \infty$, this theorem is also valid and the formulation is much simpler.

Problem (39) – (41) becomes well-posed provided that the points $\{b_j\}$ belong to Γ and the corresponding existence and uniqueness hold (see [52]). In this case we can consider more general parabolic operator than that at the beginning of this section. We consider the equation (see [52])

$$Mu = u_t - Lu = f(t,x),\quad (t,x) \in Q = (0,T) \times G, \tag{52}$$

where $Lu = \sum_{i,j=1}^n a_{ij}(t,x)u_{x_i x_j} + \sum_{i=1}^n a_i(t,x)u_{x_i} + a_0(t,x)u$, $G \in \mathbb{R}^n$ is a bounded domain with a boundary Γ. The initial-boundary conditions are as follows:

$$Bu|_S = g(t,x)\ \ (S = (0,T)\times\Gamma),\quad u|_{t=0} = u_0(x), \tag{53}$$

where $Bu = \sum_{i,j=1}^n a_{ij}(t,x)\nu_i \frac{\partial u}{\partial x_i} + \beta(t,x)u$, ν is the ourward unit normal to Γ. The overdetemination conditions agree with (41) where $b_i \in \Gamma$, $\{b_i\}_{i=1}^r$ is a set of points. The problem is to find a solution to equation (39) satisfying (41), (53) and the unknown function $g(t,x) = \sum_{j=1}^r \alpha_i(t)\Phi_i(t,x)$, where Φ_i are given and α_i are unknowns. As before, a parameter $\delta > 0$ is admissible if $\overline{B_\delta(b_i)} \cap \overline{B_\delta(b_j)} = \emptyset$ for $i \neq j$, $i,j = 1,2,\ldots,r$. The operator L is elliptic, i. e., there exists a constant $\delta_0 > 0$ such that

$$\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \delta_0|\xi|^2\ \forall \xi \in \mathbb{R}^n,\ \forall (t,x) \in Q.$$

We also assume that

$$a_i \in L_p(Q),\ a_{kl} \in C(\overline{Q}),\ \beta \in W_p^{s_0,2s_0}(S),\ a_{kl}|_\Gamma \in W_p^{s_0,2s_0}(S), \tag{54}$$

$$a_i \in L_\infty(0,T;W_p^1(G_\delta)),\ a_{kl} \in L_\infty(0,T;W_\infty^1(G_\delta)), \tag{55}$$

where $i = 0,1\ldots,n,\ k,l = 1,\ldots,n$. We use the straightening the boundary, i. e., the transformation $z_n = y_n - \gamma(y')$, $z' = y'$, where y is a local coordinate system at b_i. We assume that $\Gamma_\delta = G_\delta \cap \Gamma \in C^3$. Put $U = \{z : |z'| < \delta, 0 < z_n < \delta_1\}$ and $B'_\delta = \{z' : |z'| < \delta\}$. Set $Q_0^\tau = (0,\tau) \times U$, $Q_0 = (0,T) \times U$ and $S_0^\tau = (0,\tau) \times B'_\delta$, $S_0 = (0,T) \times B'_\delta$.

$$u_0(x) \in W_p^{2-\frac{2}{p}}(G),\ f \in L_p(Q), \tag{56}$$

$$g(0,x) = B(x,0,\partial_x)u_0|_\Gamma,\ g \in W_p^{s_0,2s_0}(S), \tag{57}$$

$$\beta \in L_p(0,T;W_p^{2-1/p}(\Gamma_\delta) \cap W_p^1(\Gamma_\delta;W_p^{1/2-1/2p}(0,T)). \tag{58}$$

Let U_i be a coordinate neighborhood of $b_i \in \Gamma$. We straighten the boundary and take the new coordinate system $z = (z', z_n)$. Next, we assume that

$$\nabla_{z'}\varphi_k g(t, x^k(z',0)) \in W_p^{s_0,2s_0}(S_0)\ (k = 1,2,\ldots,r), \tag{59}$$

$$\nabla_{z'}\varphi_i f(t,x^i(z)) \in L_p(Q_0),\ \nabla_{z'}\varphi_i u_0(x^i(z)) \in W_p^{2-2/p}(U),$$
$$\nabla_{z'}a_{kl}(t,x^i(z',0)) \in W_p^{s_0,2s_0}(S_0)\ \ (k,l = 1,2\ldots,n,\ i \le r). \tag{60}$$

These conditions (60), (59) are independent of the local coordinate systems y and z.

Next, we assume that

$$\Phi_i \in W_p^{s_0,2s_0}(S),\ \nabla_{z'}\Phi_i(t,x^j(z',0)) \in W_p^{s_0,2s_0}(S_0),\ i,j = 1,2,\ldots,r. \tag{61}$$

Introduce the matrix $\Phi(t)$ with the entries $\phi_{ij} = \Phi_j(t,b_i)$ $(i,j = 1,2,\ldots,r)$ and suppose that

$$|\det \Phi| \ge \delta_1 > 0\ \forall t \in [0,T]. \tag{62}$$

$$\psi_i \in W_p^{s_1}(0,T),\ u_0(b_i) = \psi_i(0)\ (i = 1,\ldots,r), \tag{63}$$

where δ_1 is a positive constant. Take the first of equalities (40) at $(0,b_j)$. We have

$$Bu_0 = \frac{\partial u_0(b_j)}{\partial N} + \beta(0,b_j)u_0(b_j) = g(0,b_j) = \sum_{i=1}^r \alpha_i(0)\Phi_i(0,b_j),\ j = 1,\ldots,r. \tag{64}$$

This system is uniquely solvable and we can find the quantities $\alpha_i(0)$. In this case we have the equality

$$\frac{\partial u_0(x)}{\partial N} + \beta(0,x)u_0(x) = g(0,x) = \sum_{i=1}^r \alpha_i(0)\Phi_i(0,x)\ \forall x \in G, \tag{65}$$

where $\alpha_j(0)$ are solutions to system (64).

Theorem 6. [52] *Let the conditions* (54) – (63), (65) *be satisfied. Then there exists a unique solution u, $\vec{\alpha}$ to problem* (39) – (41) *such that* $u \in W_p^{1,2}(Q)$, $\vec{\alpha} \in W_p^{s_0,2s_0}(0,T)$, $\nabla_{z'}\varphi_i u(x^i(z)) \in W_p^{1,2}(Q_0),\ i = 1,\ldots,r,$

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ОБ ОБРАТНЫХ ЗАДАЧАХ С ТОЧЕЧНЫМ ПЕРЕОПРЕДЕЛЕНИЕМ ДЛЯ МАТЕМАТИЧЕСКИХ МОДЕЛЕЙ ТЕПЛОМАССОПЕРЕНОСА

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Данная работа – обзор, посвященный обратным задачам восстановления источников и коэффициентов (параметров среды) в математических моделях тепломассопереноса. Главное внимание уделяется вопросам корректности обратных задач с точечными условиями переопределения. Такие вопросы возникают в теории тепломассопереноса, в задачах окружающей среды и экологии, при описании процессов диффузии, фильтрации и во многих других областях. Примерами могут служить задача определения тензора теплопроводности или задача определения источников загрязнения в водном бассейне или атмосфере. Мы рассматриваем три типа задач. Первая из них – задача восстановления точечных или распределенных источников. Описываются условия существования и единственности решений, приводятся примеры неединственности и, в модельных ситуациях, даются оценки на число замеров, которые позволяет полностью определить интенсивности источников и их местоположение. Вторая задача – задача восстановления параметров среды например, теплопроводности. Третья задача – задача о восстановлении граничных режимов, т.е. потока через поверхность или коэффициента теплопередачи.

Ключевые слова: тепломассоперенос; математическое моделирование; параболическое уравнение; обратная задача; единственность; точечный источник.

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