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ANALYSIS OF BIHARMONIC AND HARMONIC MODELS BY THE METHODS OF ITERATIVE EXTENSIONS

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The article describes the results of recent years on the analysis of biharmonic and harmonic models by the methods of iterative extensions. In mechanics, hydrodynamics and heat engineering, various stationary physical systems are modeled using boundary value problems for inhomogeneous Sophie Germain and Poisson equations. Deflection of plates, flows during fluid flows are described using the biharmonic model, i.e. boundary value problem for the inhomogeneous Sophie Germain equation. Deflection of membranes, stationary temperature distributions near the plates are described using the harmonic model, i.e. boundary value problem for the inhomogeneous Poisson equation. With the help of the developed methods of iterative extensions, efficient algorithms for solving the problems under consideration are obtained.

Keywords: biharmonic and harmonic models; methods of iterative extensions.

Dedicated to the anniversary of Alexander Leonidovich Shestakov

Introduction

First, we consider the biharmonic model, i.e. mixed boundary value problem for the inhomogeneous biharmonic equation

$$\Delta^2 \breve{u} = \breve{f} \tag{1}$$

in a bounded domain on the plane $\Omega \subset \mathrm{R}^2$ with the boundary conditions of four types

$$\breve{u} = \frac{\partial \breve{u}}{\partial n}\,|_{\Gamma_0} = 0,\ \ \breve{u} = l_1 \breve{u}\,|_{\Gamma_1} = 0,$$

$$\frac{\partial \breve{u}}{\partial n} = l_1 \breve{u}\,|_{\Gamma_1} = 0,\ \ l_1 \breve{u} = l_2 \breve{u}\,|_{\Gamma_3} = 0,$$

where

$$\partial\Omega = \bar{s},\ \ s = \Gamma_0 \bigcup \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3,\ \ \Gamma_i \bigcap \Gamma_j = \emptyset,\ \ i \neq j,\ \ i,j = 0,1,2,3,$$

$$l_1 \breve{u} = \Delta \breve{u} + (1-\sigma) n_1 n_2 \breve{u}_{xy} - n_2^2 \breve{u}_{xx} - n_1^2 \breve{u}_{yy},$$

$$l_2 \breve{u} = \frac{\partial \Delta \breve{u}}{\partial n} + (1-\sigma)\frac{\partial}{\partial s}(n_1 n_2 (\breve{u}_{yy} - \breve{u}_{xx}) + (n_1^2 - n_2^2)\breve{u}_{xy}),$$

$$n_1 = -\cos(n,x),\ \ n_2 = -\cos(n,y),\ \ \sigma \in (0;\,1).$$

The biharmonic model can be formulated as a scalar model, i.e. the problem of representing a functional in the form of a dot product

$$\breve{u} \in \breve{H}:\ \ [\breve{u},\breve{v}] = F(\breve{v})\ \forall \breve{v} \in \breve{H},\ \ F \in \breve{H}', \tag{2}$$

where the Sobolev space is

$$\breve{H}=\breve{H}(\Omega)=\left\{\breve{v}\in W_2^2(\Omega):\ \breve{v}|_{\Gamma_0\bigcup\Gamma_1}=0,\ \frac{\partial\breve{v}}{\partial n}|_{\Gamma_0\bigcup\Gamma_2}=0\right\},$$

the bilinear form, i.e. the dot product, is

$$[\breve{u},\breve{v}]=\Lambda(\breve{u},\breve{v})=\int\limits_{\Omega}(\sigma\Delta\breve{u}\Delta\breve{v}+(1-\sigma)(\breve{u}_{xx}\breve{v}_{xx}+2\breve{u}_{xy}\breve{v}_{xy}+\breve{u}_{yy}\breve{v}_{yy}))d\Omega,\ \sigma\in(0;\,1).$$

If $\breve{f}$ is a given function, then the functional

$$F\left(\breve{v}\right)=(\breve{u},\breve{v})=\int\limits_{\Omega}\breve{f}\breve{v}d\Omega.$$

For problem (2), the following assumption ensures the existence and uniqueness of its solution [1, 4]

$$\exists c_1,c_2\in(0;+\infty):\ c_1\left\|\breve{v}\right\|^2_{W_2^2(\Omega)}\leq\Lambda(\breve{v},\breve{v})\leq c_2\left\|\breve{v}\right\|^2_{W_2^2(\Omega)}\ \ \forall\breve{v}\in\breve{H}.$$

Second, we consider the harmonic model, i.e. mixed boundary value problem for the inhomogeneous harmonic equation

$$-\Delta\breve{u}=\breve{f}\tag{3}$$

in a bounded domain on the plane $\Omega\subset\mathrm{R}^2$ with the boundary conditions of two types

$$\breve{u}\,|_{\Gamma_1}=0\ ,$$

$$\frac{\partial\breve{u}}{\partial n}\,|_{\Gamma_2}=0,$$

where

$$\partial\Omega=\bar{s},\ \ s=\Gamma_1\bigcup\Gamma_2,\ \ \Gamma_1\bigcap\Gamma_2=\emptyset.$$

The harmonic model can be formulated as a scalar model, the problem of representing a functional in the form of a dot product

$$\breve{u}\in\breve{H}:\ [\breve{u},\breve{v}]=F(\breve{v})\ \forall\breve{v}\in\breve{H},\ \ F\in\breve{H}',\tag{4}$$

where the Sobolev space is

$$\breve{H}=\breve{H}(\Omega)=\left\{\breve{v}\in W_2^1(\Omega):\ \ \breve{v}|_{\Gamma_1}=0\right\},$$

the bilinear form, i.e. the dot product, is

$$[\breve{u},\breve{v}]=\mathrm{A}(\breve{u},\breve{v})=\int\limits_{\Omega}(\breve{u}_x\breve{v}_x+\breve{u}_y\breve{v}_y)d\Omega\,.$$

If $\breve{f}$ is a given function, then the linear functional

$$F\left(\breve{v}\right)=(\breve{u},\breve{v})=\int\limits_{\Omega}\breve{f}\breve{v}d\Omega\,.$$

For problem (4), the following assumption ensures the existence and uniqueness of its solution [1, 4]

$$\exists c_1, c_2 \in (0; +\infty): \ c_1 \|\breve{v}\|^2_{W_2^1(\Omega)} \leq \mathrm{A}(\breve{v}, \breve{v}) \leq c_2 \|\breve{v}\|^2_{W_2^1(\Omega)} \ \forall \breve{v} \in \breve{H}.$$

Within the framework of the considered direction, such problems were studied by the fictitious domain methods, for example, in works by A.M. Matsokin, S.V. Nepomnyashchikh [3], S.B. Sorokin [5], G.I. Marchuk, Yu.A. Kuznetsov, A.M. Matsokin [2] and others. There are difficulties in solving the above problems. The promising direction of the fictitious domain methods for solving these problems also has difficulties. We use the fact that if the problems considered as systems are similar, then they have similar properties, and the methods for solving these problems are also similar to each other. To develop new efficient methods, we use generalizations of the fictitious domain method, i.e. methods of iterative extensions. In the fictitious domain method, on the example of mechanics, we increase the support reaction and the stiffness of the material on a fictitious continuation, i.e. additionally we use the choice of two parameters. Let us minimize the error in a norm stronger than the energy norm of the emerging problem. We apply the method of minimal residuals with indication of the conditions sufficient for its convergence. With this new approach, the relative errors of the proposed iterative processes are dominated by infinitely decreasing geometric progressions. The main goal of the described works is the development of asymptotically optimal methods for solving the above problems [6–12].

1. Analysis of Biharmonic Model

1.1. Biharmonic Model

Let us present the problem to be solved for $\omega = 1$ and the fictitious problem for $\omega = \mathrm{II}$

$$\breve{u}_\omega \in \breve{H}_\omega: \ \Lambda_\omega(\breve{u}_\omega, \breve{v}_\omega) = F_\omega(\breve{v}_\omega) \ \forall \breve{v}_\omega \in \breve{H}_\omega, \ F_\omega \in \breve{H}'_\omega, \tag{5}$$

where we use Sobolev spaces

$$\breve{H}_\omega = \breve{H}_\omega(\Omega_\omega) = \left\{ \breve{v}_\omega \in W_2^2(\Omega_\omega): \ \breve{v}_\omega|_{\Gamma_{\omega,0} \bigcup \Gamma_{\omega,1}} = 0, \ \frac{\partial \breve{v}_\omega}{\partial n_\omega}|_{\Gamma_{\omega,0} \bigcup \Gamma_{\omega,2}} = 0 \right\}$$

in the bounded domains $\Omega_\omega \subset \mathrm{R}^2$ with the boundaries

$$\partial\Omega_\omega = \bar{s}_\omega, \ s_\omega = \Gamma_{\omega,0} \bigcup \Gamma_{\omega,1} \bigcup \Gamma_{\omega,2} \bigcup \Gamma_{\omega,3},$$

$$\Gamma_{\omega,i} \bigcap \Gamma_{\omega,j} = \emptyset, if i \neq j, i, j = 0, 1, 2, 3$$

n_ω are outer normals to $\partial\Omega_\omega$, bilinear forms at $\breve{a}_\omega \in [0; +\infty)$, $\sigma_\omega \in (0; 1)$ are

$$\Lambda_\omega(\breve{u}_\omega, \breve{v}_\omega) = \int\limits_{\Omega_\omega} (\sigma_\omega \Delta\breve{u}_\omega \Delta\breve{v}_\omega + (1 - \sigma_\omega)(\breve{u}_{\omega xx}\breve{v}_{\omega xx} + 2\breve{u}_{\omega xy}\breve{v}_{\omega xy} + \breve{u}_{\omega yy}\breve{v}_{\omega yy}) + a_\omega \breve{u}_\omega \breve{v}_\omega) d\Omega_\omega.$$

Each of the problems in (5) has a unique solution under the assumptions [1, 4]

$$\exists c_1, c_2 \in (0; +\infty): \ c_1 \|\breve{v}_\omega\|^2_{W_2^2(\Omega_\omega)} \leq \Lambda_\omega(\breve{v}_\omega, \breve{v}_\omega) \leq c_2 \|\breve{v}_\omega\|^2_{W_2^2(\Omega_\omega)} \ \forall \breve{v}_\omega \in \breve{H}_\omega.$$

If $\breve{f}_\omega$ is a given function, then

$$F_\omega\left(\breve{v}_\omega\right) = \int\limits_{\Omega_\omega} \breve{f}_\omega \breve{v}_\omega d\Omega_\omega.$$

In the problem to be solved, with $\omega = 1,\ a_1 = 0,\ \Gamma_{1,0} \neq \emptyset$. In the fictitious problem with $\omega = \mathrm{II},\ \breve{f}_{\mathrm{II}} = 0,\ \breve{u}_{\mathrm{II}} = 0$.

1.2. Continued Biharmonic Model and Its Analytical Study

Let us present the continued problem

$$\breve{u} \in \breve{V}:\ \ \Lambda_1(\breve{u}, I_1\breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}, \breve{v}) = F_1(I_1\breve{v})\ \forall \breve{v} \in \breve{V}, \tag{6}$$

where we use the extended solution space

$$\breve{V} = \breve{V}(\Pi) = \left\{ \breve{v} \in W_2^2(\Pi):\ \ \breve{v}|_{\Gamma_0 \bigcup \Gamma_1} = 0,\ \left.\frac{\partial \breve{v}}{\partial n}\right|_{\Gamma_0 \bigcup \Gamma_2} = 0 \right\}.$$

We assume that the solution domain of the original problem is complemented to the rectangle

$$\bar{\Omega}_1 \bigcup \bar{\Omega}_{\mathrm{II}} = \bar{\Pi},\ \Omega_1 \bigcap \Omega_{\mathrm{II}} = \emptyset,\ \Omega_1,\ \Omega_{\mathrm{II}} \subset \mathrm{R}^2,$$

and the boundary of the rectangular domain is

$$\partial\Pi = \bar{s},\ \ s = \Gamma_1 \bigcup \Gamma_2,\ \ \Gamma_1 \bigcap \Gamma_2 = \emptyset.$$

We assume that the boundaries of the first domain and the second domain intersect each other

$$\partial\Omega_1 \bigcap \partial\Omega_{\mathrm{II}} = \bar{S},\ \ S = \Gamma_{1,0} \bigcap \Gamma_{\mathrm{II},3} \neq \emptyset,$$

n is an outer normal to $\partial\Pi$. Subspace of solutions of the continued problem is

$$\breve{V}_1 = \breve{V}_1(\Pi) = \left\{ \breve{v}_1 \in \breve{V}:\ \ \breve{v}_1|_{\Pi\Omega_1} = 0 \right\}.$$

In the formulation of the continued problem, we use the projection operator

$$I_1: \breve{V} \mapsto \breve{V}_1,\ \ \breve{V}_1 = \mathrm{im} I_1,\ \ I_1 = I_1^2.$$

We introduce subspaces

$$\breve{V}_3 = \breve{V}_3(\Pi) = \left\{ \breve{v}_3 \in \breve{V}:\ \ \breve{v}_3|_{\Pi\Omega_{\mathrm{II}}} = 0 \right\},\ \breve{V}_0 = \breve{V}_1 \oplus \breve{V}_3,$$

$$\breve{V}_2 = \breve{V}_2(\Pi) = \left\{ \breve{v}_2 \in \breve{V}:\ \Lambda(\breve{v}_2, \breve{v}_0) = 0\ \forall \breve{v}_0 \in \breve{V}_0 \right\},$$

$$\breve{V} = \breve{V}_1 \oplus \breve{V}_2 \oplus \breve{V}_3 = \breve{V}_1 \oplus \breve{V}_{\mathrm{II}},\ \ \breve{V}_{\mathrm{I}} = \breve{V}_1 \oplus \breve{V}_2,\ \ \breve{V}_{\mathrm{II}} = \breve{V}_2 \oplus \breve{V}_3.$$

Direct sums are considered using the inner product generated by the bilinear form

$$\Lambda(\breve{u}, \breve{v}) = \Lambda_1(\breve{u}, \breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}, \breve{v})\ \forall \breve{u}, \breve{v} \in \breve{V}.$$

It is assumed that the bilinear form is such that

$$\exists c_1, c_2 > 0: \ c_1 \|\breve{v}\|^2_{W^2_2(\Pi)} \leq \Lambda(\breve{v},\breve{v}) \leq c_2 \|\breve{v}\|^2_{W^2_2(\Pi)} \ \forall \breve{v} \in \breve{V}.$$

We use the statement on the possibility to continue the functions

$$\exists \breve{\beta}_1 \in (0;1], \ \breve{\beta}_2 \in [\breve{\beta}_1;1]: \ \breve{\beta}_1 \Lambda(\breve{v}_2,\breve{v}_2) \leq \Lambda_{\mathrm{II}}(\breve{v}_2,\breve{v}_2) \leq \breve{\beta}_2 \Lambda(\breve{v}_2,\breve{v}_2) \ \forall \breve{v}_2 \in \breve{V}_2.$$

Note that

$$\breve{H}_\omega(\Omega_\omega) = \breve{V}_\omega(\Omega_\omega), \ \omega \in \{1, \ \mathrm{II}\}.$$

Statement 1. [12] *The solution to problem (6)* $\breve{u} \in \breve{V}_1$ *coincides with the solution to problem (3) for* $\omega = 1$ *on* Ω_1 *and equals zero on* Ω_{II}.

The study of the continued biharmonic model is carried out by the modified method of fictitious components [6, 7, 9, 10]:

$$\breve{u}^k \in \breve{V}: \ \Lambda(\breve{u}^k - \breve{u}^{k-1}, \breve{v}) = -\tau_{k-1}(\Lambda_1(\breve{u}^{k-1}, I_1\breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}^{k-1}, \breve{v}) - F_1(I_1\breve{v})) \ \forall \breve{v} \in \breve{V},$$

$$\tau_0 = 1, \ \tau_{k-1} = \tau = 2/(\breve{\beta}_1 + \breve{\beta}_2), \ k \in \mathrm{N}\setminus\{1\}, \ \forall \breve{u}^0 \in \breve{V}_1 \subset \breve{V}. \quad (7)$$

Let us introduce the norm

$$\|\breve{v}\|_{\breve{V}} = \sqrt{\Lambda(\breve{v},\breve{v})}.$$

Theorem 1. [12] *There exist the following convergence estimates:*

$$\left\|\breve{u}^k - \breve{u}\right\|_{\breve{V}} \leq \varepsilon \left\|\breve{u}^0 - \breve{u}\right\|_{\breve{V}}, \ k \in \mathrm{N},$$

where

$$\varepsilon = \delta_1 q^{k-1}, \ \delta_1 = \sqrt{\|I_1\|^2_{\breve{V}} - 1}, \ 0 \leq q = (\breve{\beta}_2 - \breve{\beta}_1)\Big/(\breve{\beta}_1 + \breve{\beta}_2) < 1.$$

1.3. Continued Biharmonic Model under Discretization and Its Numerical Analysis

Let us discretize the continued model when

$$\Pi = (0;b_1) \times (0;b_2), \ \Gamma_1 = \{b_1\} \times (0;b_2) \bigcup (0;b_1) \times \{b_2\},$$

$$\Gamma_2 = \{0\} \times (0;b_2) \bigcup (0;b_1) \times \{0\}, \ b_1, b_2 \in (0;+\infty).$$

Let us introduce the grid

$$(x_i;y_j) = ((i-1,5)h_1;(j-1,5)h_2),$$

$$h_1 = b_1/(m-1,5), \ h_2 = b_2/(n-1,5), \ i = 1,2...,m, \ j = 1,2...,n, \ m-2, n-2 \in \mathrm{N}.$$

We consider grid functions at the grid nodes

$$v_{i,j} = v(x_i;y_j) \in \mathrm{R}, \ i = 1,2...,m, \ j = 1,2...,n, \ m-2, n-2 \in \mathrm{N}.$$

Use the completion for the grid functions

$$\Phi^{i,j}(x;y) = \Psi^{1,i}(x)\Psi^{2,j}(y), \ i = 2,...,m-1, \ j = 2,...,n-1, \ m-2, n-2 \in \mathrm{N},$$

$$\Psi^{1,i}(x) = [2/i]\Psi(x/h_1 - i + 4) + \Psi(x/h_1 - i + 3) - [(i+1)/m]\Psi(x/h_1 - i + 1),$$

$$\Psi^{2,j}(y) = [2/j]\Psi(y/h_2 - j + 4) + \Psi(y/h_2 - j + 3) + [(j+1)/n]\Psi(y/h_2 - j + 1),$$

$$\Psi(z) = \begin{cases} 0,5z^2, & z \in [0;1], \\ -z^2 + 3z - 1,5, & z \in [1;2], \\ 0,5z^2 - 3z - 4,5, & z \in [2;3], \\ 0, & z \notin (0;3). \end{cases}$$

We consider the basis functions to be equal to zero outside the rectangle:

$$\Phi^{i,j}(x;y) = 0,\ (x;y) \notin \Pi,\ i = 2...,m-1,\ j = 2...,n-1,\ m-2, n-2 \in \mathrm{N}.$$

Linear combinations of basis functions give a finite-dimensional subspace in the extended space

$$\tilde{V} = \left\{ \tilde{v} = \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} v_{i,j} \Phi^{i,j}(x;y) \right\} \subset \breve{V}.$$

Consider the continued model in the matrix form

$$\bar{u} \in \mathrm{R}^N:\ \ B\bar{u} = \bar{f},\ \bar{f} \in \mathrm{R}^N, \tag{8}$$

under the assumption that the projection operator vanishes the coefficients of the basis functions whose carriers do not belong entirely to the first domain, and the continued matrix and the continued right-hand side of the system are defined by the equalities

$$\langle B\bar{u}, \bar{v} \rangle = \Lambda_1(\tilde{u}, I_1\tilde{v}) + \Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \tilde{V},\ \langle \bar{f}, \bar{v} \rangle = F_1(I_1\tilde{v})\ \forall \tilde{v} \in \tilde{V},$$

$$\langle \bar{f}, \bar{v} \rangle = (\bar{f}, \bar{v})h_1h_2 = \bar{f}\,\bar{v}h_1h_2,\ \bar{v} = (v_1, v_2, ..., v_N)' \in \mathrm{R}^N,\ N = (m-2)(n-2).$$

In this case, we enumerate first the coefficients of the basis functions with carriers that belong entirely to the inside of the first domain. Next, we enumerate the coefficients of the basis functions with the carriers that cross the boundary of both the first and second domains. We finish the enumeration with the coefficients of the basis functions with carriers that belong entirely to the inside of the second domain. Then the vectors have the following structure

$$\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')',\ \bar{u} = (\bar{u}_1', \bar{0}', \bar{0}')',\ \ \bar{f} = (\bar{f}_1', \bar{0}', \bar{0}')'.$$

The matrix has the structure

$$B = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix}.$$

We define the matrices

$$\langle \Lambda_{\mathrm{I}}\bar{u}, \bar{v} \rangle = \Lambda_1(\tilde{u}, \tilde{v}),\ \langle \Lambda_{\mathrm{II}}\bar{u}, \bar{v} \rangle = \Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

The matrices have the structure

$$\Lambda_{\mathrm{I}} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{21} & \Lambda_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \Lambda_{\mathrm{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix}.$$

Define the extended matrix

$$\Lambda = \Lambda_{\mathrm{I}} + \Lambda_{\mathrm{II}} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{21} & \Lambda_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix}.$$

We introduce the corresponding subspaces

$$\bar{V}_1 = \left\{ \bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N : \ \bar{v}_2 = \bar{0}, \ \bar{v}_3 = \bar{0} \right\},$$

$$\bar{V}_3 = \left\{ \bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N : \ \bar{v}_1 = \bar{0}, \ \bar{v}_2 = \bar{0} \right\}, \ \bar{V}_0 = \bar{V}_1 \oplus \bar{V}_3,$$

$$\bar{V}_2 = \left\{ \bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N : \ \Lambda_{11}\bar{v}_1 + \Lambda_{12}\bar{v}_2 = \bar{0}, \ \Lambda_{32}\bar{v}_2 + \Lambda_{33}\bar{v}_3 = \bar{0} \right\}.$$

There exist decompositions

$$\mathrm{R}^N = \bar{V}_1 \oplus \bar{V}_2 \oplus \bar{V}_3 = \bar{V}_1 \oplus \bar{V}_{\mathrm{II}}, \ \bar{V}_{\mathrm{I}} = \bar{V}_1 \oplus \bar{V}_2, \ \bar{V}_{\mathrm{II}} = \bar{V}_2 \oplus \bar{V}_3.$$

Let us present the assumptions about the continuation in the matrix form

$$\exists \beta_1 \in (0; +\infty), \ \beta_2 \in [\beta_1; +\infty) : \ \beta_1 \langle \Lambda\bar{v}_2, \bar{v}_2 \rangle \le \langle \Lambda_{\mathrm{II}}\bar{v}_2, \bar{v}_2 \rangle \le \beta_2 \Lambda \langle \Lambda\bar{v}_2, \bar{v}_2 \rangle \ \forall \bar{v}_2 \in \bar{V}_2.$$

The matrix form of the continued biharmonic model is

$$B\bar{u} = \bar{f}, \ \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix}.$$

The original problem in the matrix form and the fictitious problem in the matrix form are

$$\Lambda_{11}\bar{u}_1 = \bar{f}_1, \ \begin{bmatrix} \Lambda_{02} & \Lambda_{23} \\ \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, \ \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}.$$

When studying the continued biharmonic model in the matrix form, we define the extended matrix in a new way as follows:

$$C = \Lambda_{\mathrm{I}} + \gamma\Lambda_{\mathrm{II}}, \ \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{21} & \Lambda_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix}, \ \gamma \in (0; +\infty).$$

We use the fulfilment of the statements about the continuation of functions in the following form:

$$\exists \gamma_1 \in (0; +\infty), \ \gamma_2 \in [\gamma_1; +\infty) : \ \gamma_1^2 \langle C\bar{v}_2, C\bar{v}_2 \rangle \le \langle \Lambda \mathrm{A}_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2 \rangle \le \gamma_2^2 \langle C\bar{v}_2, C\bar{v}_2 \rangle \ \forall \bar{v}_2 \in \bar{V}_2,$$

$$\exists \alpha \in (0; +\infty) : \ \langle \Lambda_{\mathrm{I}}\bar{v}_2, \Lambda_{\mathrm{I}}\bar{v}_2 \rangle \le \alpha^2 \langle \Lambda_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2 \rangle \ \forall \bar{v}_2 \in \bar{V}_2.$$

To solve problem (8), as a generalization of the modified method of fictitious components, we apply the method of iterative extensions [8, 9, 11,12]:

$$\bar{u}^k \in \mathrm{R}^N : \ C(\bar{u}^k - \bar{u}^{k-1}) = -\tau_{k-1}(B\bar{u}^{k-1} - \bar{f}), \ k \in \mathrm{N}, \tag{9}$$

$$\forall \bar{u}^0 \in \bar{V}_1, \ \gamma > \alpha, \ \tau_0 = 1, \ \tau_{k-1} = \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1} \right\rangle \big/ \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1} \right\rangle, \ k \in \mathrm{N}\{1\},$$

where residuals, corrections, and equivalent residuals are respectively calculated

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f},\ \bar{w}^{k-1} = C^{-1}\bar{r}^{k-1},\ \bar{\eta}^{k-1} = B\bar{w}^{k-1},\ k \in \mathrm{N}.$$

Let us define the norm

$$\|\bar{v}\|_{C^2} = \sqrt{\langle C^2\bar{v}, \bar{v}\rangle}\ \forall \bar{v} \in \mathrm{R}^N.$$

Theorem 2. [14] *Process (9) has the following estimate*

$$\left\|\bar{u}^k - \bar{u}\right\|_{C^2} \le \varepsilon \left\|\bar{u}^0 - \bar{u}\right\|_{C^2},\ \varepsilon = 2(\gamma_2/\gamma_1)(\alpha/\gamma)^{k-1},\ k \in \mathrm{N}.$$

Let us present an algorithmic implementation of the method of iterative extensions for the biharmonic model. We use the method of minimal residuals to solve problem (8).

I. Set the initial approximation and the iterative parameter

$$\forall \bar{u}^0 \in \bar{V}_1,\ \tau_0 = 1.$$

II. Calculate the residual

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f},\ k \in \mathrm{N}.$$

III. Calculate the absolute error norm squared

$$\mathrm{E}_{k-1} = \left\langle \bar{r}^{k-1}, \bar{r}^{k-1} \right\rangle,\ k \in \mathrm{N}.$$

IV. Find the correction

$$\bar{w}^{k-1}:\ C\bar{w}^{k-1} = \bar{r}^{k-1},\ k \in \mathrm{N}.$$

V. Calculate the equivalent residual

$$\bar{\eta}^{k-1} = B\bar{w}^{k-1},\ k \in \mathrm{N}\left\{1\right\}.$$

VI. Calculate the iteration parameter

$$\tau_{k-1} = \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1} \right\rangle / \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1} \right\rangle,\ k \in \mathrm{N}\left\{1\right\}.$$

VII. Calculate the next approximation

$$\bar{u}^k = \bar{u}^{k-1} - \tau_{k-1}\bar{w}^{k-1},\ k \in \mathrm{N}.$$

VIII. Check the iteration stop criterion

$$\mathrm{E}_{k-1} \le \mathrm{E}_0\mathrm{E}^2,\ k \in \mathrm{N}\left\{1\right\},\ \mathrm{E} \in (0;\ 1).$$

2. Analysis of Harmonic Model

2.1. Harmonic Model

Let us present the problem to be solved for $\omega = 1$ and the fictitious problem for $\omega = \mathrm{II}$

$$\breve{u}_\omega \in \breve{H}_\omega:\ \mathrm{A}_\omega(\breve{u}_\omega, \breve{v}_\omega) = F_\omega(\breve{v}_\omega)\ \forall \breve{v}_\omega \in \breve{H}_\omega,\ F_\omega \in \breve{H}'_\omega, \tag{10}$$

where we use the Sobolev spaces

$$\breve{H}_\omega = \breve{H}_\omega(\Omega_\omega) = \left\{\breve{v}_\omega \in W_2^1(\Omega_\omega) : \ \breve{v}_\omega|_{\Gamma_{\omega,1}} = 0\right\}$$

in the bounded domains $\Omega_\omega \subset \mathrm{R}^2$ with the boundaries

$$\partial\Omega_\omega = \bar{s}_\omega, \ s_\omega = \Gamma_{\omega,1}\bigcup\Gamma_{\omega,2}, \Gamma_{\omega,1}\bigcap\Gamma_{\omega,2} = \emptyset,$$

n_ω are outer normals to$\partial\Omega_\omega$, bilinear forms at $\kappa_\omega \in [0; +\infty)$ are

$$\mathrm{A}_\omega(\breve{u}_\omega, \breve{v}_\omega) = \int\limits_{\Omega_\omega} (\breve{u}_{\omega x}\breve{v}_{\omega x} + \breve{u}_{\omega y}\breve{v}_{\omega y} + \kappa_\omega \breve{u}_\omega \breve{v}_\omega) d\Omega_\omega.$$

Each of the problems in (10) has a unique solution under the assumptions [1, 4]

$$\exists c_1, c_2 \in (0; +\infty) : \ c_1 \left\|\breve{v}_\omega\right\|^2_{W_2^1(\Omega_\omega)} \leq \mathrm{A}_\omega(\breve{v}_\omega, \breve{v}_\omega) \leq c_2 \left\|\breve{v}_\omega\right\|^2_{W_2^1(\Omega_\omega)} \ \forall \breve{v}_\omega \in \breve{H}_\omega,$$

If $\breve{f}_\omega$ is a given function, then

$$F_\omega\left(\breve{v}_\omega\right) = \int\limits_{\Omega_\omega} \breve{f}_\omega \breve{v}_\omega d\Omega_\omega.$$

In the problem to be solved with $\omega = 1, \ \kappa_1 = 0, \ \Gamma_{1,1} \neq \emptyset$. In the fictitious problem with $\omega = \mathrm{II}, \ \breve{f}_{\mathrm{II}} = 0, \ \breve{u}_{\mathrm{II}} = 0$.

2.2. Continued Harmonic Model and Its Analytical Study

Let us present the continued problem

$$\breve{u} \in \breve{V} : \ \ \mathrm{A}_1(\breve{u}, I_1\breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v}) = F_1(I_1\breve{v}) \ \forall \breve{v} \in \breve{V} \tag{11}$$

where we use the extended solution space

$$\breve{V} = \breve{V}(\Pi) = \left\{\breve{v} \in W_2^1(\Pi) : \ \breve{v}|_{\Gamma_1} = 0\right\}.$$

We assume that the solution domain of the original problem is complemented to the rectangle

$$\bar{\Omega}_1\bigcup\bar{\Omega}_{\mathrm{II}} = \bar{\Pi}, \ \Omega_1\bigcap\Omega_{\mathrm{II}} = \emptyset, \ \Omega_1, \, \Omega_{\mathrm{II}} \subset \mathrm{R}^2,$$

and the boundary of the rectangular domain is

$$\partial\Pi = \bar{s}, \ s = \Gamma_1\bigcup\Gamma_2, \ \Gamma_1\bigcap\Gamma_2 = \emptyset.$$

We assume that the boundaries of the first domain and the second domain intersect each other

$$\partial\Omega_1\bigcap\partial\Omega_{\mathrm{II}} = \bar{S}, \ S = \Gamma_{1,1}\bigcap\Gamma_{\mathrm{II},2} \neq \emptyset,$$

n is an outer normal to $\partial\Pi$. The subspace of solutions to the continued problem is

$$\breve{V}_1 = \breve{V}_1(\Pi) = \left\{\breve{v}_1 \in \breve{V} : \ \breve{v}_1|_{\Pi\Omega_1} = 0\right\}.$$

In the formulation of the continued problem, we use the projection operator

$$I_1 : \breve{V} \mapsto \breve{V}_1,\ \breve{V}_1 = imI_1,\ I_1 = I_1^2.$$

We introduce subspaces

$$\breve{V}_3 = \breve{V}_3(\Pi) = \left\{ \breve{v}_3 \in \breve{V} :\ \ \breve{v}_3|_{\Pi\Omega_{\mathrm{II}}} = 0 \right\},\ \breve{V}_0 = \breve{V}_1 \oplus \breve{V}_3,$$

$$\breve{V}_2 = \breve{V}_2(\Pi) = \left\{ \breve{v}_2 \in \breve{V} :\ \mathrm{A}(\breve{v}_2, \breve{v}_0) = 0\ \forall \breve{v}_0 \in \breve{V}_0 \right\},$$

$$\breve{V} = \breve{V}_1 \oplus \breve{V}_2 \oplus \breve{V}_3 = \breve{V}_1 \oplus \breve{V}_{\mathrm{II}},\ \breve{V}_{\mathrm{I}} = \breve{V}_1 \oplus \breve{V}_2,\ \breve{V}_{\mathrm{II}} = \breve{V}_2 \oplus \breve{V}_3.$$

Direct sums are considered using the dot product generated by the bilinear form

$$\mathrm{A}(\breve{u}, \breve{v}) = \mathrm{A}_1(\breve{u}, \breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}, \breve{v})\ \forall \breve{u}, \breve{v} \in \breve{V}.$$

It is assumed that the bilinear form is such that

$$\exists c_1, c_2 > 0 :\ c_1 \left\| \breve{v} \right\|^2_{W_2^1(\Pi)} \leq \mathrm{A}(\breve{v}, \breve{v}) \leq c_2 \left\| \breve{v} \right\|^2_{W_2^1(\Pi)}\ \ \forall \breve{v} \in \breve{V}.$$

We use the statement on the possibility of continuing the functions

$$\exists \breve{\beta}_1 \in (0;1],\ \breve{\beta}_2 \in [\breve{\beta}_1;1] :\ \breve{\beta}_1 \mathrm{A}(\breve{v}_2, \breve{v}_2) \leq \mathrm{A}_{\mathrm{II}}(\breve{v}_2, \breve{v}_2) \leq \breve{\beta}_2 \mathrm{A}(\breve{v}_2, \breve{v}_2)\ \forall \breve{v}_2 \in \breve{V}_2.$$

Note that

$$\breve{H}_\omega(\Omega_\omega) = \breve{V}_\omega(\Omega_\omega),\ \omega \in \{1,\ \mathrm{II}\}.$$

Statement 2. [12] *The solution to the problem (11)* $\breve{u} \in \breve{V}_1$ *coincides with the solution to problem (10) for* $\omega = 1$ *on* Ω_1 *and equals to zero on* Ω_{II}.

The study of the continued harmonic model is carried out by the modified method of fictitious components [6, 8, 9]:

$$\breve{u}^k \in \breve{V} :\ \mathrm{A}(\breve{u}^k - \breve{u}^{k-1}, \breve{v}) = -\tau_{k-1}(\mathrm{A}_1(\breve{u}^{k-1}, I_1\breve{v}) + \mathrm{A}_{\mathrm{II}}(\breve{u}^{k-1}, \breve{v}) - F_1(I_1\breve{v}))\ \forall \breve{v} \in \breve{V},$$

$$\tau_0 = 1,\ \tau_{k-1} = \tau = 2/(\breve{\beta}_1 + \breve{\beta}_2),\ k \in \mathrm{N} \backslash \{1\},\ \forall \breve{u}^0 \in \breve{V}_1 \subset \breve{V}. \tag{12}$$

Let us introduce the norm

$$\left\| \breve{v} \right\|_{\breve{V}} = \sqrt{\mathrm{A}(\breve{v}, \breve{v})}\,.$$

Theorem 3. [12] *There exist the following convergence estimates:*

$$\left\| \breve{u}^k - \breve{u} \right\|_{\breve{V}} \leq \varepsilon \left\| \breve{u}^0 - \breve{u} \right\|_{\breve{V}},\ k \in \mathrm{N},$$

where

$$\varepsilon = \delta_1 q^{k-1},\ \delta_1 = \sqrt{\left\| I_1 \right\|^2_{\breve{V}} - 1},\ 0 \leq q = (\breve{\beta}_2 - \breve{\beta}_1) \Big/ (\breve{\beta}_1 + \breve{\beta}_2) < 1.$$

2.3. Continued Harmonic Model under Discretization and Its Numerical Analysis

Let us discretize the continued model when

$$\Pi = (0; b_1) \times (0; b_2),\ \Gamma_1 = \{b_1\} \times (0; b_2) \bigcup (0; b_1) \times \{b_2\},$$

$$\Gamma_2 = \{0\} \times (0; b_2) \bigcup (0; b_1) \times \{0\},\ b_1, b_2 \in (0; +\infty).$$

Let us introduce the grid

$$(x_i; y_j) = ((i - 1,5)h_1; (j - 1,5)h_2),$$

$$h_1 = b_1/(m - 1,5),\ h_2 = b_2/(n - 1,5),\ i = 1, 2..., m,\ j = 1, 2..., n,\ m - 2, n - 2 \in \mathrm{N}.$$

We consider the grid functions at the grid nodes

$$v_{i,j} = v(x_i; y_j) \in \mathrm{R},\ i = 1, 2..., m,\ j = 1, 2..., n,\ m - 2, n - 2 \in \mathrm{N}.$$

Use completion for the grid functions

$$\Phi^{i,j}(x; y) = \Psi^{1,i}(x)\Psi^{2,j}(y),\ i = 2..., m - 1,\ j = 2..., n - 1,\ m - 2, n - 2 \in \mathrm{N},$$

$$\Psi^{1,i}(x) = [2/i]\Psi(x/h_1 - i + 3,5) + \Psi(x/h_1 - i + 2,5),$$

$$\Psi^{2,j}(y) = [2/j]\Psi(y/h_2 - j + 3,5) + \Psi(y/h_2 - j + 2,5),$$

$$\Psi(z) = \begin{cases} z, & z \in [0; 1], \\ 2 - z, & z \in [1; 2], \\ 0, & z \notin (0; 2). \end{cases}$$

We define the basis functions to be equal to zero outside the rectangle:

$$\Phi^{i,j}(x; y) = 0,\ (x; y) \notin \Pi,\ i = 2..., m - 1,\ j = 2..., n - 1,\ m - 2, n - 2 \in \mathrm{N}.$$

Linear combinations of basis functions give a finite-dimensional subspace in the extended space

$$\tilde{V} = \left\{ \tilde{v} = \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} v_{i,j} \Phi^{i,j}(x; y) \right\} \subset \breve{V}.$$

Consider the continued model in the matrix form

$$\bar{u} \in \mathrm{R}^N :\ \ B\bar{u} = \bar{f},\ \bar{f} \in \mathrm{R}^N, \tag{13}$$

under the assumption that the projection operator vanishes the coefficients of the basis functions whose carriers do not belong entirely to the first domain, and the continued matrix and the continued right-hand side of the system are defined by the equalities

$$\langle B\bar{u}, \bar{v}\rangle = \mathrm{A}_1(\tilde{u}, I_1\tilde{v}) + \mathrm{A}_{\Pi}(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \tilde{V},\ \langle \bar{f}, \bar{v}\rangle = F_1(I_1\tilde{v})\ \forall \tilde{v} \in \tilde{V},$$

$$\langle \bar{f}, \bar{v}\rangle = (\bar{f}, \bar{v})h_1h_2 = \bar{f}\,\bar{v}h_1h_2,\ \bar{v} = (v_1, v_2, ..., v_N)' \in \mathrm{R}^N,\ N = (m - 2)(n - 2).$$

In this case, we enumerate first the coefficients of the basis functions with carriers that belong entirely to the inside of the first domain. Next, we enumerate the coefficients of the basis functions with the carriers that cross the boundary of both the first and second domains. We finish the enumeration with the coefficients of the basis functions with carriers that belong entirely to the inside of the second domain. Then the vectors have the following structure

$$\bar{v} = (\bar{v}_1',\ \bar{v}_2',\ \bar{v}_3')',\ \bar{u} = (\bar{u}_1',\ \bar{0}',\ \bar{0}')',\ \ \bar{f} = (\bar{f}_1',\ \bar{0}',\ \bar{0}')'.$$

The matrix has the structure

$$B = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}.$$

We define the matrices

$$\langle \mathrm{A}_{\mathrm{I}}\bar{u}, \bar{v}\rangle = \mathrm{A}_1(\tilde{u}, \tilde{v}),\ \langle \mathrm{A}_{\mathrm{II}}\bar{u}, \bar{v}\rangle = \mathrm{A}_{\mathrm{II}}(\tilde{u}, \tilde{v})\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

The matrices have the structure

$$\mathrm{A}_{\mathrm{I}} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathrm{A}_{\mathrm{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}.$$

Define the extended matrix

$$\mathrm{A} = \mathrm{A}_{\mathrm{I}} + \mathrm{A}_{\mathrm{II}} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}.$$

We introduce the corresponding subspaces

$$\bar{V}_1 = \left\{\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N :\ \bar{v}_2 = \bar{0},\ \bar{v}_3 = \bar{0}\right\},$$

$$\bar{V}_3 = \left\{\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N :\ \bar{v}_1 = \bar{0},\ \bar{v}_2 = \bar{0}\right\},\ \bar{V}_0 = \bar{V}_1 \oplus \bar{V}_3,$$

$$\bar{V}_2 = \left\{\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N :\ \mathrm{A}_{11}\bar{v}_1 + \mathrm{A}_{12}\bar{v}_2 = \bar{0},\ \mathrm{A}_{32}\bar{v}_2 + \mathrm{A}_{33}\bar{v}_3 = \bar{0}\right\}.$$

There exist decompositions

$$\mathrm{R}^N = \bar{V}_1 \oplus \bar{V}_2 \oplus \bar{V}_3 = \bar{V}_1 \oplus \bar{V}_{\mathrm{II}},\ \bar{V}_{\mathrm{I}} = \bar{V}_1 \oplus \bar{V}_2,\ \bar{V}_{\mathrm{II}} = \bar{V}_2 \oplus \bar{V}_3.$$

Let us present the assumptions about the continuation in the matrix form

$$\exists \beta_1 \in (0; +\infty),\ \beta_2 \in [\beta_1; +\infty) :\ \beta_1 \langle \mathrm{A}\bar{v}_2, \bar{v}_2\rangle \le \langle \mathrm{A}_{\mathrm{II}}\bar{v}_2, \bar{v}_2\rangle \le \beta_2 \langle \mathrm{A}\bar{v}_2, \bar{v}_2\rangle\ \ \forall \bar{v}_2 \in \bar{V}_2.$$

The matrix form of the continued harmonic model is

$$B\bar{u} = \bar{f},\ \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix}.$$

The original problem in the matrix form and the fictitious problem in the matrix form are

$$\mathrm{A}_{11}\bar{u}_1 = \bar{f}_1,\ \begin{bmatrix} \mathrm{A}_{02} & \mathrm{A}_{23} \\ \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix},\ \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}.$$

When studying the continued harmonic model in the matrix form, we define the extended matrix in a new way as follows:

$$C = \mathrm{A}_{\mathrm{I}} + \gamma \mathrm{A}_{\mathrm{II}},\ \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & 0 \\ \mathrm{A}_{21} & \mathrm{A}_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathrm{A}_{02} & \mathrm{A}_{23} \\ 0 & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix},\ \gamma \in (0; +\infty).$$

We use the fulfilment of the statements about the continuation of functions in the following form:

$$\exists \gamma_1 \in (0; +\infty),\ \gamma_2 \in [\gamma_1; +\infty) :\ \gamma_1^2 \langle C\bar{v}_2, C\bar{v}_2\rangle \le \langle \mathrm{A}_{\mathrm{II}}\bar{v}_2, \mathrm{A}_{\mathrm{II}}\bar{v}_2\rangle \le \gamma_2^2 \langle C\bar{v}_2, C\bar{v}_2\rangle\ \ \forall \bar{v}_2 \in \bar{V}_2,$$

$$\exists\alpha \in (0;+\infty): \ \langle \mathrm{A_I}\bar{v}_2, \mathrm{A_I}\bar{v}_2\rangle \le \alpha^2 \langle \mathrm{A_{II}}\bar{v}_2, \mathrm{A_{II}}\bar{v}_2\rangle \ \ \forall \bar{v}_2 \in \bar{V}_2.$$

To solve problem (13), as a generalization of the modified method of fictitious components, we apply the method of iterative extensions [8, 9, 11,12]:

$$\bar{u}^k \in \mathrm{R}^N: \ C(\bar{u}^k - \bar{u}^{k-1}) = -\tau_{k-1}(B\bar{u}^{k-1} - \bar{f}), \ k \in \mathrm{N}, \tag{14}$$

$$\forall \bar{u}^0 \in \bar{V}_1, \ \gamma > \alpha, \ \tau_0 = 1, \ \tau_{k-1} = \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1}\right\rangle / \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1}\right\rangle, \ k \in \mathrm{N}\left\{1\right\},$$

where residuals, corrections, and equivalent residuals are respectively calculated as

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f}, \ \bar{w}^{k-1} = C^{-1}\bar{r}^{k-1}, \ \bar{\eta}^{k-1} = B\bar{w}^{k-1}, \ k \in \mathrm{N}.$$

Let us define the norm

$$\|\bar{v}\|_{C^2} = \sqrt{\langle C^2\bar{v}, \bar{v}\rangle} \ \ \forall \bar{v} \in \mathrm{R}^N.$$

Theorem 4. [15] *Process (14) has the following estimate:*

$$\left\|\bar{u}^k - \bar{u}\right\|_{C^2} \le \varepsilon \left\|\bar{u}^0 - \bar{u}\right\|_{C^2}, \ \varepsilon = 2(\gamma_2/\gamma_1)(\alpha/\gamma)^{k-1}, \ k \in \mathrm{N}.$$

Let us present an algorithmic implementation of the method of iterative extensions for the biharmonic model. We use the method of minimal residuals to solve problem (13).

I. Set the initial approximation and the iterative parameter

$$\forall \bar{u}^0 \in \bar{V}_1, \ \tau_0 = 1.$$

II. Calculate the residual

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f}, \ k \in \mathrm{N}.$$

III. Calculate the absolute error norm squared

$$\mathrm{E}_{k-1} = \left\langle \bar{r}^{k-1}, \bar{r}^{k-1}\right\rangle, \ k \in \mathrm{N}.$$

IV. Find the correction

$$\bar{w}^{k-1}: \ C\bar{w}^{k-1} = \bar{r}^{k-1}, \ k \in \mathrm{N}.$$

V. Calculate the equivalent residual

$$\bar{\eta}^{k-1} = B\bar{w}^{k-1}, \ k \in \mathrm{N}\left\{1\right\}.$$

VI. Calculate the iteration parameter

$$\tau_{k-1} = \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1}\right\rangle / \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1}\right\rangle, \ k \in \mathrm{N}\left\{1\right\} .$$

VII. Calculate the next approximation

$$\bar{u}^k = \bar{u}^{k-1} - \tau_{k-1}\bar{w}^{k-1}, \ k \in \mathrm{N}.$$

VIII. Check the iteration stop criterion

$$\mathrm{E}_{k-1} \le \mathrm{E}_0\mathrm{E}^2, \ k \in \mathrm{N}\left\{1\right\}, \ \mathrm{E} \in (0; \ 1).$$

Conclusion

The biharmonic and harmonic problems considered as models and systems are similar, have similar properties and similar methods for their solution. With necessary changes, the corresponding results for the biharmonic and harmonic models hold for the scalar model.

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References

1. Aubin J.-P. *Approximation of Elliptic Boundary-Value Problems.* New York, Wiley-Interscience, 1972.
2. Bank R.E., Rose D.J. Marching Algorithms for Elliptic Boundary Value Problems. I: the Constant Coefficient Case. *SIAM Journal on Numerical Analysis*, 1977, vol. 14, no. 5, pp. 792–829.
3. Marchuk G.I., Kuznetsov Yu.A., Matsokin A.M. Fictitious Domain and Domain Decomposion Methods. *Russian Journal Numerical Analysis and Mathematical Modelling*, 1986, vol. 1, no. 1, pp. 3–35.
4. Swarztrauber P.N. A Direct Method for Discrete Solution of Separable Elliptic Equations. *SIAM Journal on Numerical Analysis*, 1974, vol. 11, no. 6, pp. 1136–1150.
5. Swarztrauber P.N. The Method of Cyclic Reduction, Fourier Analysis and FACR Algorithms for the Discrete Solution of Poisson's Equations on a Rectangle. *SIAM Review*, 1977, vol. 19, no. 3, pp. 490–501.
6. Manteuffel T. An Incomlete Factorization Technigue for Positive Definite Linear Systems. *Mathematics of Computation*, 1980, vol. 38, no. 1, pp. 114–123.
7. Matsokin A.M., Nepomnyaschikh S.V. The Fictitious-Domain Method and Explicit Continuation Operators. *Computational Mathematics and Mathematical Physics*, 1993, vol. 33, no. 1, pp. 52–68.
8. Mukanova B. Numerical Reconstruction of Unknown Boundary Data in the Cauchy Problem for Laplace's Equation. *Inverse Problems in Science and Engineering*, 2012, vol. 21, no. 8, pp. 1255–1267. DOI: 10.1080/17415977.2012.744405
9. Oganesyan L.A., Rukhovets L.A. *Variation-Difference Methods for solving Elliptic Equations* [Variatsionno-raznostnye metody resheniya ellipticheskikh uravnenii]. Erevan AN ArmSSR, 1979. (in Russian)
10. Sorokin S.B. Analytical Solution of Generalized Spectral Problem in the Method of Recalculating Boundary Conditions for a Biharmonic Equation. *Siberian Journal Numerical Mathematics*, 2013, vol. 16, no. 3, pp. 267–274. DOI: 10.1134/S1995423913030063
11. Sorokin S.B. An Efficient Direct Method for the Numerical Solution to the Cauchy Problem for the Laplace Equation. *Numerical Analysis and Applications*, 2019, vol. 12, no. 12, pp. 87–103. (in Russian) DOI: 10.1134/S1995423919010075
12. Ushakov A.L. About Modeelling of Deformations of Plates. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2015, vol. 8, no. 2, pp. 138–142. (in Russian) DOI: 10.14529/mmp150213
13. Ushakov A.L. Investigation of a Mixed Boundary Value Problem for the Poisson Equation. *International Russian Automation Conference*, Sochi, Russian Federation, 2020, article ID: 9208198, 6 p. DOI: 10.1109/RusAutoCon49822.2020.9208198
14. Ushakov A.L. Numerical Anallysis of the Mixed Boundary Value Problem for the Sophie Germain Equation. *Journal of Computational and Engineering Mathematics*, 2021, vol. 8, no. 1, pp. 46–59. DOI: 10.14529/jcem210104

15. Ushakov A.L. Analysis of the Mixed Boundary Value Problem for the Poisson's Equation. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2021, vol. 13, no. 1, pp. 29–40. (in Russian) DOI: 10.14529/mmph210104
16. Ushakov A.L. Research of the Boundary Value Problem for the Sophie Germain Equationinin in a Cyber-Physical System. *Studies in Systems, Decision and Control*, 2021, vol. 338, pp. 51–63. DOI: 10.1007/978-3-030-66077-2-5
17. Ushakov A.L. Analysis of the Boundary Value Problem for the Poisson Equation. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2022, vol. 14, no. 1, pp. 64–76. DOI: 10.14529/mmph220107
18. Ushakov A.L. Anallysis of the Problem for the Biharmonic Equation. *Journal of Computational and Engineering Mathematics*, 2022, vol. 9, no. 1, pp. 43–58. DOI: 10.14529/jcem220105

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АНАЛИЗ БИГАРМОНИЧЕСКИХ И ГАРМОНИЧЕСКИХ МОДЕЛЕЙ МЕТОДАМИ ИТЕРАЦИОННЫХ РАСШИРЕНИЙ

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В статье приводится описание результатов за последние годы по анализу бигармонических и гармонических моделей методами итерационных расширений. Различные стационарные физические системы в механике, гидродинамике, теплотехнике моделируются с помощью краевых задач для неоднородных уравнений Софи Жермен и Пуассона. Используя бигармоническую модель, т.е. краевую задачу для неоднородного уравнения Софи Жермен, описывают прогибание пластин, потоки при течениях жидкостей. Используя гармоническую модель, т.е. краевую задачу для неоднородного уравнения Пуассона, описывают прогибания мембран, стационарные распределения температур у пластин. С помощью разработанных методов итерационных расширений получаются эффективные алгоритмы решения рассматриваемых задач.

Ключевые слова: бигармонические и гармонические модели; методы итерационных расширений.

Литература

1. Aubin, J.-P. Approximation of Elliptic Boundary-Value Problems / J.-P. Aubin. – New York: Wiley-Interscience, 1972.
2. Bank, R.E. Marching Algorithms for Elliptic Boundary Value Problems. I: the Constant Coefficient Case / R.E. Bank, D.J. Rose // SIAM Journal on Numerical Analysis. – 1977. – V. 14, № 5. – P 792–829.
3. Marchuk, G.I. Fictitious Domain and Domain Decomposion Methods / G.I. Marchuk, Yu.A. Kuznetsov, A.M. Matsokin // Russian Journal Numerical Analysis and Mathematical Modelling. – 1986. – V. 1, № 1. – P. 3–35.
4. Swarztrauber, P.N. A Direct Method for Discrete Solution of Separable Elliptic Equations / P.N Swarztrauber // SIAM Journal on Numerical Analysis. – 1974. – V. 11, № 6. – P. 1136–1150.

5. Swarztrauber, P.N. The Method of Cyclic Reduction, Fourier Analysis and FACR Algorithms for the Discrete Solution of Poisson's Equations on a Rectangle / P.N Swarztrauber // SIAM Review. – 1977. – V. 19, № 3. – P. 490–501.
6. Manteuffel, T. An Incomlete Factorization Technigue for Positive Definite Linear Systems / T. Manteuffel // Mathematics of Computation. – 1980. – V. 38, № 1. – P. 114–123.
7. Matsokin, A.M. The Fictitious-Domain Method and Explicit Continuation Operators / A.M. Matsokin, S.V. Nepomnyaschikh // Computational Mathematics and Mathematical Physics. – 1993. – V. 33, № 1. – P. 52–68.
8. Mukanova, B. Numerical Reconstruction of Unknown Boundary Data in the Cauchy Problem for Laplace's Equation / B. Mukanova // Inverse Problems in Science and Engineering. – 2012. – V. 21, № 8. – P. 1255–1267.
9. Оганесян, Л.А. Вариационно-разностные методы решения эллиптических уравнений / Л.А. Оганесян, Л.А. Руховец. – Ереван: Издательство АН Армянской ССР, 1979.
10. Sorokin, S.B. Analytical Solution of Generalized Spectral Problem in the Method of Recalculating Boundary Conditions for a Biharmonic Equation / S.B. Sorokin // Siberian Journal Numerical Mathematics. – 2013. – V. 16, № 3. – P. 267–274.
11. Сорокин, С.Б. Экономичный прямой метод численного решения задачи Коши для уравнения Лапласа / С.Б. Сорокин // Сибирский журнал вычислительной математики. – 2019. – Т. 12, № 12. – С. 87–103.
12. Ушаков, А.Л. О моделировании деформаций пластин / А.Л. Ушаков // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2015. – Т. 8, № 2. – С. 138–142.
13. Ushakov, A.L. Investigation of a Mixed Boundary Value Problem for the Poisson Equation / A.L. Ushakov // International Russian Automation Conference. – 2020. – Article ID: 9208198. – 6 с.
14. Ushakov, A.L. Numerical Anallysis of the Mixed Boundary Value Problem for the Sophie Germain Equation / A.L. Ushakov // Journal of Computational and Engineering Mathematicsю – 2021. – V. 8, № 1. – P. 46–59.
15. Ушаков, А.Л. Анализ смешанной краевой задачи для уравнения Пуассона / А.Л. Ушаков // Вестник ЮУрГУ. Серия: Математика, Механика, Физика. – 2021. – Т. 13, № 1. – С. 29–40.
16. Ushakov, A.L. Research of the Boundary Value Problem for the Sophie Germain Equationinin in a Cyber-Physical System / A.L. Ushakov // Studies in Systems, Decision and Control. – 2021. – V. 338. – P. 51–63.
17. Ushakov, A.L. Analysis of the Boundary Value Problem for the Poisson Equation / A.L. Ushakov // Вестник ЮУрГУ. Серия: Математика. Механика. Физика. – 2022. – Т. 14, № 1. – С. 64–76.
18. Ushakov, A.L. Anallysis of the Problem for the Biharmonic Equation / A.L. Ushakov // Journal of Computational and Engineering Mathematics. – 2022. – V. 9, № 1. – P. 43–58.

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