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A NEW FORMULA ON THE CONJUGATE GRADIENT METHOD FOR REMOVING IMPULSE NOISE IMAGES

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A variety of conjugate gradient algorithms are constructed on the coefficient conjugate. In this paper, a new coefficient conjugate based on the quadratic model for impulse noise removal is proposed. Its global convergence results might be achieved under Wolfe line search circumstances. To demonstrate the performance of the conjugate gradient approach for impulse noise reduction, numerical experiments are provided.

Keywords: image processing; impulse noise; conjugate gradient method; global convergence.

Introduction

Due to their enormous scale, many real-world applications result in non-linear optimization issues. As a result, first-order approaches are the most popular. Gradient techniques present one of the most successful first-order ways for handling both unconstrained and restricted issues in image processing.

A two-phase approach is suggested in the paper [1], which combines the benefits of adaptive median filter and variational method. The adaptive median filter is utilized in the first phase for salt-and-pepper noise [2]. Let X be the true image with M-by-N pixels, $A = 1, 2, 3, ..., M \times 1, 2, 3, ..., N$ be the index set of X, and $N \subset A$ be the set of indices of the noise pixels discovered in the first phase. The issue is to discover a practical approach to reduce the functional as follows:

$$f_\alpha(u) = \sum_{(i,j)\in N} \left[|u_{i,j} - y_{i,j}| + \frac{\beta}{2}(2 \times S^1_{i,j} + S^2_{i,j}) \right], \tag{1}$$

where β is the regularization parameter, $S^1_{i,j} = 2\sum_{(m,n)\in P_{i,j}\cap N^c} \phi_\alpha(u_{i,j} - y_{m,n}), S^2_{i,j} = \sum_{(m,n)\in P_{i,j}\cap N} \phi_\alpha(u_{i,j} - y_{m,n})$ and the edge-preserving potential function is $\varphi_\alpha = \sqrt{a + x^2}, \alpha > 0$. Let $P_{i,j}$ be the collection of four closest neighbors of a pixel at the point $(i, j) \in A, y_{i,j}$ be the observed pixel value of the image at the position (i, j), and $u_{i,j} = \lfloor u_{i,j} \rfloor_{(i,j)\in N}$ be a lexicographically ordered column vector of length. Because of the term $|u_{i,j} - y_{i,j}|$, the functional of problem (1) is nonsmooth. It is generally assumed that this nonsmooth term can be removed from (1) because the term keeps the minimizer u near the original image y so that the pixels, uncorrupted in the original image, are not altered. Then, the following smooth functional is obtained:

$$f_\alpha(u) = \sum_{(i,j)\in N} \lfloor (2 \times S^1_{i,j} + S^2_{i,j}) \rfloor. \tag{2}$$

One of the most helpful iterative approaches for removing impulse noise is the conjugate gradient (CG) method:

$$f(u^*) = \min_{x\in R^N} f(u), \tag{3}$$

where $f : R^n \longrightarrow R$ is a continuously differentiable function, see the paper [3]. In each iteration, the CG methods only need to determine the gradient of the goal function. As a result, using this strategy to handle large-scale unconstrained optimization issues is very desirable. The iterative formula of these approaches is as follows:

$$u_{k+1} = u_k + \alpha_k d_k. \tag{4}$$

If f is convex quadratic, then the one-dimensional minimizer along the ray $u_k + \alpha_k d_k$ may be calculated analytically as follows:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k}. \tag{5}$$

For general non-linear functions, it is necessary to use an iterative procedure. More details can be found in the paper [4].

The Wolfe requirements are frequently employed in the convergence analysis and implementation of conjugate gradient algorithms to identify the step length α_k satisfying:

$$f_{k+1} \leq f_k + \delta \alpha_k g_k^T d_k, \tag{6}$$

$$\left| d_k^T g_{k+1} \right| \leq -\sigma\ d_k^T g_k, \tag{7}$$

where $0 < \delta < \sigma < 1$. More details can be found in the paper [5].

In practical implementations, the conjugate gradient search direction for the next iteration has the following form:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{8}$$

where β_k is a scalar. Two famous ways of choosing β_k are:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \beta_{k+1}^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k}. \tag{9}$$

These were given by Fletcher–Reeves (FR) method [6], and the Dai – Yuan (DY) method [7] independently. These methods have certain strong convergence characteristics, but their numerical results are weaker than those for other methods [4].

Over the years, several variations of this technique were developed, and some are now in use. For instance, Hideaki and Yasushi [8] and Basim [9], which are as follows:

$$\beta_k^{HY} = \frac{\|g_{k+1}\|^2}{2/\alpha_k (f_k - f_{k+1})}, \beta_k^{B} = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/\alpha_k - g_k^T d_k/2}. \tag{10}$$

The numerical efficiency of these algorithms is also very high. For unconstrained problems, the quadratic model was documented to boost efficiency, in order to optimize the benefits of the original conjugate gradient approaches, which primarily consider objective function information and demonstrate global convergence. There exist different methods that we can see in [10–14].

In this study, using a conjugacy condition, we present a modified conjugate gradient method. Independent of the line search and the convexity assumption on the objective function, the new search direction satisfies the descent property enough. Taking common assumptions into account, the proposed method provides global convergence for generic functions. The new approach appears to be superior in terms of removing impulsive noise, according to numerical tests.

1. Deriving New Coefficient Conjugate

As we know, when the objective function to be reduced is a convex quadratic and accurate line searches are employed, the following results are obtained:

$$d_{k+1}^T Q d_k = 0, \tag{11}$$

where Q is Hessian matrix. This is called the conjugacy condition. Since for the conjugate gradient method the search direction is calculated as $d_{k+1} = -g_{k+1} + \beta_k d_k$, we have

$$\beta_k = \frac{g_{k+1}^T Q s_k}{d_k^T Q s_k}. \tag{12}$$

thus a coefficient conjugacy is obtained, see [8].

Now, to derive the new conjugate gradient method, we use a quadratic model, which can be written as:

$$f_{k+1} = f_k + s_k^T g_k + \frac{1}{2} s_k^T Q(u_k) s_k. \tag{13}$$

Calculate the first order derivative and equate to zero:

$$\nabla f_{k+1} = g_k + Q(u_k) s_k. \tag{14}$$

The best second-order curvature is

$$s_k^T Q(u_k) s_k = (f_k - f_{k+1}) + 1/2 s_k^T y_k. \tag{15}$$

Putting equation (15) in (12), we obtain a new formula for β_k as follows:

$$\beta_k = \frac{g_{k+1}^T y_k}{(f_k - f_{k+1})/\alpha_k + 1/2 d_k^T y_k}. \tag{16}$$

We can write (16) if the step length is exact:

$$\beta_k = \frac{g_{k+1}^T y_k}{(f_k - f_{k+1})/\alpha_k + 1/2 g_k^T g_k}. \tag{17}$$

Furthermore, if f is a quadratic function and accurate line search is used, the following results are obtained:

$$\beta_k = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/\alpha_k + 1/2 d_k^T y_k} \tag{18}$$

and

$$\beta_k = \frac{\|g_{k+1}\|^2}{(f_k - f_{k+1})/\alpha_k + 1/2 g_k^T g_k}. \tag{19}$$

Therefore, our proposition becomes obvious, the so-called BDC and BSC.

On the basis of the above, a new algorithm for the suggested technique is offered.

1.1. Algorithms for BDC and BSC

Step 1. Let u_1 be an initial point. Set $k = 1$ and $d_1 = -g_1$.
Step 2. Let α_k be the stepsize that satisfies the Wolfe criteria (6) and (7).
Step 3. Let $x_{k+1} = x_k + \alpha_k d_k$. If criteria terminate is satisfied, then stop.
Step 4. Update β_k by formula (18) then generate d_{k+1} by (15).
Step 5. Set $k = k + 1$ and go to Step 2.

It reveals a good property for new formula conjugate gradient method it is descent condition. Provide, in this study additional property are very important in the convergence proving. The following theorem summarize the above discussion.

Theorem 1. *If we create $\{x_k\}$ and $\{d_k\}$ using the new approach, we get:*

$$d_{k+1}^T g_{k+1} < 0 \text{ and } d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k. \tag{20}$$

Proof. Obviously, if $d_k = -g_k$ then $d_1^T g_1 < 0$. Suppose that $d_k^T g_k < 0$ for any k. It is easy to get from (8) and (19) that:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1} = -\beta_k((f_k - f_{k+1})/\alpha_k + 1/2 d_k^T y_k) + \beta_k d_k^T g_{k+1}. \tag{21}$$

We may deduce the following from the preceding equation:

$$d_{k+1}^T g_{k+1} = \beta_k[d_k^T g_{k+1} - ((f_k - f_{k+1})/\alpha_k + 1/2 d_k^T y_k)] \tag{22}$$

again also from equation (16) and equation (21), we get:

$$d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k \tag{23}$$

It is clear that $d_k^T g_k < 0$, then we obtain :

$$d_{k+1}^T g_{k+1} < 0 \tag{24}$$

The proof is completed. In a similar way a descent property of a BSC method is proven. □

2. Convergence Analysis

The convergence of the novel algorithm is investigated. On the objective function, we assume the following assumptions.

1. The level set "$D = \{u/f(u) \le f(u_0)\}$" is bounded, where x_0 is the starting point.
2. The gradient is Lipschitz continuous in some neighborhood D, which contains L_0; i.e. there exists L such that:

$$\|g(\upsilon) - g(\omega)\| \le L \|\upsilon - \omega\|, \forall \upsilon, \omega \in D \tag{25}$$

For more details see [7, 15].

Zoutendijk [16] arrived at the following significant conclusion.

Lemma 1. *Suppose that all assumptions hold. Consider any iteration technique that uses the Wolfe line search to retrieve α_k. Then:*

$$\sum_{k=1}^{8} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < 8. \tag{26}$$

Theorem 2. *Suppose that all assumptions hold. If the formula for β_k satisfies (21), then there exists*

$$\lim_{k \longrightarrow \infty} \inf \|g_k\| = 0. \tag{27}$$

Proof. By induction, suppose that (27) is not true. Rewriting (8) as $d_{k+1} + g_{k+1} = \beta_k d_k$, and squaring both sides, we get:

$$\|d_{k+1}\|^2 + \|g_{k+1}\|^2 + 2d_{k+1}^T g_{k+1} = (\beta_k)^2 \|d_k\|^2. \tag{28}$$

Applying (23) yields:

$$\|d_{k+1}\|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2} \|d_k\|^2 - 2d_{k+1}^T g_{k+1} - \|g_{k+1}\|^2. \tag{29}$$

By dividing (29) by $(d_{k+1}^T g_{k+1})^2$ on both sides, we get:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} = \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \frac{\|g_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}}. \tag{30}$$

By completing the square, the equation becomes:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} \leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \left(\frac{\|g_{k+1}\|}{(d_{k+1}^T g_{k+1})} + \frac{1}{\|g_{k+1}\|^2}\right) + \frac{1}{\|g_{k+1}\|^2} \leq \\ \leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} + \frac{1}{\|g_{k+1}\|^2}. \tag{31}$$

Hence,

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} = \sum_{i=1}^{k+1} \frac{1}{\|g_i\|^2}. \tag{32}$$

Assume that there exists some $c_1 > 0$ such that $\|g_k\| \geq c_1$ for all $k \in n$. Then:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{c_1^2}. \tag{33}$$

From the assumption and equation (33) we see that:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \infty. \tag{34}$$

Based on Lemma 1, we get that $\lim\limits_{k \longrightarrow \infty} \inf \|g_k\| = 0$ holds. It is easy to test that for BSC method.

□

3. Numerical Results

This section compares the new coefficient's numerical performance for salt-and-pepper impulse noise to its conventional FR approach. There are 512 × 512 gray level photos for each of the test photograph. The performance ratio is calculated as follows:

$$PSNR = 10\log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2}. \tag{35}$$

PSNR is defined as the pixel values of the restored image and the original image, respectively, where $u_{i,j}^r$ and $u_{i,j}^*$ signify the pixel values of the restored and original images, respectively (peak signal to noise ratio). The performance profile shown in Table 1 compares the performance of each of the coefficients in terms of number of iterations (NOI), function evaluations (NOF), and PSNR. If there is an inequity, we stop iterating.

$$\frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \quad \text{and} \quad \|f(u_k)\| \leq 10^{-4}(1 + |f(u_k)|) \tag{36}$$

are satisfied. Inequality (33) for stop the iteration can be seen in [17, 18].

These findings demonstrate that the new approach may effectively repair damaged image (see Figure).

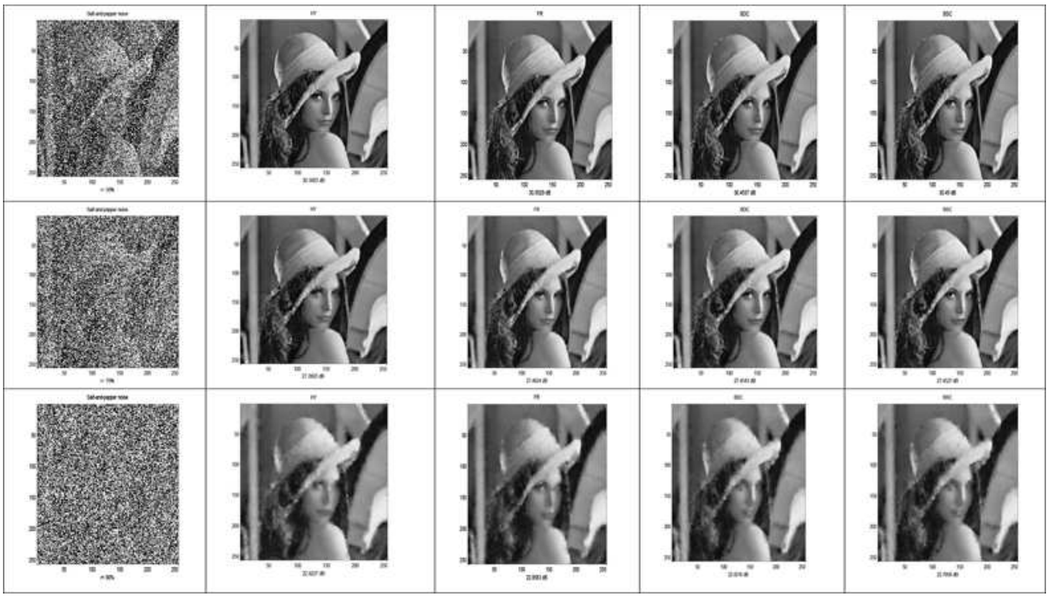
Conclusions

We present a new descent conjugate gradient method and establish its global convergence under the strong Wolfe line search conditions. Numerical results show that

Table

The numerical results for FR, HY, BDC and BSC methods

Image	Noise level r (%)	FR-Method			HY-Method			BDC-Method			BSC-Method		
		NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)
Le	50	82	153	30.5529	59	119	30.5883	39	82	30.4537	33	60	30.45
	70	81	155	27.4824	63	125	27.3603	46	96	27.4543	33	66	27.4527
	90	108	211	22.8583	92	183	22.9227	48	97	22.8216	76	148	22.7966
ho	50	52	53	34.6845	35	69	35.2505	28	59	34.8032	23	46	34.7961
	70	63	116	31.2564	41	81	30.9434	32	64	31.1492	35	72	31.1528
	90	111	214	25.287	63	125	24.9523	45	92	25.0674	63	125	25.0268
El	50	35	36	33.9129	28	53	33.9439	25	47	33.8898	25	48	33.9055
	70	38	39	31.864	31	59	31.8524	28	51	31.8235	30	56	31.8088
	90	65	114	28.2019	49	94	28.3382	37	70	28.1959	49	95	28.2139
c512	50	59	87	35.5359	41	83	35.2530	28	60	35.4474	31	66	35.4342
	70	78	142	30.6259	46	93	30.5735	36	77	30.6919	41	86	30.6545
	90	121	236	24.9362	76	157	24.8479	44	92	24.966	72	145	24.9591



From left to right: 50,70 and 90% noise, FR method, HY method, BDC and BSC methods for 256×256 Lena image

the new methods can very improve the iterations and function evaluations for impulse denoising while obtaining the same restored image quality.

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НОВАЯ ФОРМУЛА СОПРЯЖЕННОГО ГРАДИЕНТНОГО МЕТОДА ДЛЯ УДАЛЕНИЯ ИЗОБРАЖЕНИЙ ИМПУЛЬСНОГО ШУМА

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Множество алгоритмов сопряженного градиента построено на сопряженном коэффициенте. В этой статье предлагается новый сопряженный коэффициент, основанный на квадратичной модели для удаления импульсного шума. Его результаты глобальной сходимости могут быть достигнуты в условиях поиска линии Вульфа. Чтобы продемонстрировать эффективность метода сопряженных градиентов для снижения импульсного шума, проводятся численные эксперименты.

Ключевые слова: обработка изображений; импульсный шум; метод сопряженных градиентов; глобальная сходимость.

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