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MATHEMATICAL MODELLING ECONOMY

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The article presents an overview of the main methods of economic modelling used in scientific research over the past twenty years. This overview does not claim to cover all areas, methods and models used in scientific research in the field of economics, since it is impossible to do within a single article. We consider mathematical modelling of only two branches of economic theory: macroeconomics and microeconomics. At the same time, we present no literature review of methods and models of research in the section of microeconomics, which take place in the tools of scientific research, but were described in the section of macroeconomics. We believe that this review is useful to scientists engaged in the indirect study of economic phenomena and processes.

Keywords: modelling; model; macroeconomics; microeconomics; sources; review.

Economy becomes a science only when it begins to rely on mathematics. Economy does not tolerate full-scale experiments, therefore, mathematical modelling becomes the main instrumental method of indirect study of economic phenomena and processes. Similarly to the existing division of economic theory into macroeconomics and microeconomics, the article provides an overview of the main methods of mathematical modelling in such a structural division.

Table 1

Mathematical modelling of macroeconomics

Method	Model	References
1. Statistical models of macroeconomics	Macroeconomic production functions	[1, 2]
	Leontief model	[3, 4]
2. Dynamic models of macroeconomics with discrete time	Dynamic Keynesian model	[5, 6]
	Samuelson–Hicks model	[7, 8]
	Dynamic Leontief model	[9, 10]
	Neumann model	[11, 12]
3. Linear dynamic systems	Linear dynamic element	[13]
	Multiplier	[14]
	Accelerator	[15]
	Inertial link	[16]
	Transmission function	[17]
	Oscillating link	[18]
4. Nonlinear dynamic systems	Dynamic element of a nonlinear dynamic system	[19]
	Nonlinear dynamic Keynesian model	[20]
	Market cycles in economy	[21]
	Optimal control to dynamic systems	[22]
	Pontryagin maximum principle	[23]
5. Small-sector nonlinear dynamic models of macroeconomics	Solow model	[24]
	Golden rule of accumulation	[25]
	One-sector model of optimal economic growth	[26]
	Three-sector model of the economy	[27]

1. In macroeconomic production functions, the economy is considered as an unstructured unit, the input of which is resources, and the output is the final product of the functioning of the economy. In this case, the resources are considered as arguments, and the final product is a function. In practice, the Cobb-Douglas production function

$$F(K,L) = A \cdot K^{\alpha} \cdot L^{1-\alpha}$$

is most often used, where $A > 0$ is a coefficient of neutral technical progress; $\alpha \in (0,1)$ is the coefficient of elasticity of output for capital, and $(1-\alpha)$ is used for labor. With time-invariant parameters of the production function, the mathematical model is static [1, 2].

In the Leontief model, the economy is structured and consists of a finite number of autonomous industries which produce one kind of product. For its production in a particular industry, its own product and products of other industries are used. The amount of products consumed to produce a unit of product of the industry under study is taken into account by means of direct cost coefficients. These coefficients do not depend on time and on scale of production. Gross outputs of industries that ensure production of the final product are determined according to the matrix of coefficients of direct costs for a given final product [3, 4].

The gross output of the i-th product for the year x_i is divided into two parts: production consumption in all industries and non-productive consumption. The net output of the i-th product is

$$x_i - \sum_{j=1}^{n} a_{ij} x_j,\ i = 1, ..., n,$$

where $\sum_{j=1}^{n} a_{ij} x_j$ is the production consumption of the i-th product by all industries. If we equate the net output of each i-th product and its final demand y_i, then we form the system of equations

$$x_i - \sum_{j=1}^{n} a_{ij} x_j = y_i,\ i = 1, ..., n,$$

which constitutes the static Leontief model.

We draw the following conclusion: static macroeconomic models are still the best tool for system analysis of resource support for the production of products and services.

2. Linear dynamic models of macroeconomics with discrete time are represented by the dynamic Keynesian model, the dynamic Samuelson-Hicks model, the dynamic Leontief model, and the Neumann model. Here, mathematical methods and research models consider the economy as a simply connected system with discrete time.

In dynamic Keynesian model, the economy is treated as a single dynamic element Y, a time-varying endogenous variable is gross domestic product (GDP). Note that GDP consists of four parts: non-productive consumption fund C; gross private domestic investment I; government expenditures on the purchase of goods and services G, and net exports E. In the model, the economy is considered to be closed, thus net exports equal zero, and government expenditures are allocated to consumption and accumulation:

$$Y = C + I.$$

The model assumes that the demand for investment goods is constant, and the demand for consumer goods in the next year is a linear function of the current year's GDP:

$$C_{t+1}^{D} = \underline{C} + cY_t,$$

where c is the lower limit of the non-productive consumption fund; $0 < c < 1$ is a marginal propensity to consume. Dynamic Keynesian model arises if we equate the planned output of end-use goods and the projected demand for them:

$$Y_{t+1} = \underline{C} + cY_t + I.$$

This model can only be used for analysis and short-term forecasting of the economy. The model is not suitable for long-term forecasting, since it does not reflect the reproduction process of capital. From a mathematical point of view, this model is a first-order linear finite-difference equation [5, 6].

At certain values of the parameters, the Samuelson-Hicks model is an oscillatory link, and in another case, it is represented by two first-order linear dynamic elements connected in a series. The difference between the Samuelson-Hicks model and the dynamic Keynesian model is the rejection of the investments constancy and their introduction as a variable part, which is proportional to the GDP growth of the current year compared to the previous year:

$$Y_{t+1} = \underline{C} + cY_t + r(Y_t - Y_{t-1}) + I,$$

where r is the acceleration coefficient, $0 < r < 1$.

From a mathematical point of view, the Samuelson–Hicks model is a second-order linear finite-difference equation. To find solutions of the dynamic model, finite-difference equations and Laurent transformations [7, 8] are used.

Leontief dynamic model of input-output balance reflects the reproduction process, thus, it is applicable to study the behavior of the economic system over sufficiently long time intervals while maintaining the technological structure [9, 10].

The Neumann model is a generalization of the Leontief model, as it allows the production of one product in different ways. The model represents n products and m methods of their production, where each j-th method is defined by the cost column-vector a_j and the output column-vector b_j per unit of process intensity:

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}, b_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix},$$

Cost and output matrices are formed from input and output vectors:

$$A = (a_1, a_2, \ldots, a_m), B = (b_1, b_2, \ldots, b_m).$$

The input coefficients a_{ij} and the output coefficients b_{ij} are non-negative. The implementation of any process requires the costs of at least one product, i.e. for each j there exists at least one i such that $a_{ij} > 0$, and each product can be produced in at least one way, i.e. for each i there exists some j such that $b_{ij} > 0$. Thus, each column of the matrix A and each row of the matrix B must have at least one positive element.

The Neumann model describes a closed economy in which the products produced in the previous production cycle (the year $t-1$) are used to produce products in the next production cycle (the year t):

$$Ax_t \leq Bx_{t-1}, x_t \geq 0, t = 1, 2, , T,$$

where $y_t = Ax_t$ is the cost vector for a given process intensity vector x_t:

$$x_t = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix},$$

$z_t = Bx_{t-1}$ is the output vector. It is assumed that the initial stock vector $Bx_0 > 0$ is given [11, 12].

Neither the Leontief model nor the Neumann model is suitable for reflecting scientific and technological progress. In addition, these models do not reflect the reproductive process. Therefore, they can only be used for operational forecasting.

We draw the following conclusion: dynamic models of macroeconomics with discrete time are not suitable for the analysis of reproduction and reflection of scientific and technological progress, but can be successfully used for operational and short-term forecasting of economic processes and phenomena.

3. The main results in the study of dynamic systems with continuous time were obtained in the study of technical systems within the framework of the theory of automatic control. The apparatus of differential equations was used as the main mathematical toolkit. Now, the obtained scientific results of research are successfully used in the economy.

A linear dynamic element of the n-th order is given by the linear differential equation [13]:

$$\sum_{j=0}^{n} a_j y^{(j)} = \sum_{i=0}^{n} b_i x^{(i)}.$$

Most often, in practice, elements of the zero order (multiplier, accelerator) and the first order (inertial link) take place. The multiplier is a linear static link given by the equation:

$$a_0 y = b_0 x \ or \ y = \alpha x, \ \alpha = \frac{b_0}{a_0},$$

where α is the amplification factor (multiplier) [14].

The accelerator is a zero-order differentiator the output of which is proportional to the input speed. For example, the investment I can be expressed in terms of the rate of GDP change as follows:

$$I = r\frac{dY}{dt},$$

where r is the acceleration coefficient, i.e. an increase in the need for investment with an increase in GDP per unit [15].

The inertial link is given by the first-order differential equation [16]:

$$a_1 \frac{dY}{dt} + a_0 y = x(t).$$

The concept of the transfer function of a dynamic element is associated with the operator method for solving a differential equation.

The transfer function of series-connected elements is the relation of the output and input images:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{Y_2(s)}{X_1(s)} = \frac{G_2(s)Y_1(s)}{X_1(s)} = G_1(s)G_2(s).$$

Thus, the transfer function of series-connected elements is equal to the product of their transfer functions [17]. The transfer function of parallel-connected elements with a summing link is equal to the sum (difference) of the transfer functions of the elements [17]:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{Y_1(s) \pm Y_2(s)}{X(s)} = G_1(s) \pm G_2(s).$$

An oscillatory link is used to model cyclical processes in the economy. The oscillatory link is given by the second-order differential equation

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = \sum_{i=0}^{n} b_i x^{(i)}(t)$$

with a negative discriminant formed by the coefficients on the left side of the equation [18]

$$a_1^2 - 4a_2 a_0 < 0.$$

We draw the following conclusion: in the analysis of linear dynamic systems, the apparatus of differential equations should be used to model cyclical processes in the economy.

4. Dynamic systems are called non-linear if they contain at least one non-linear element.

The method of analysis of a nonlinear system depends on the type of nonlinearity. There are two main approaches: direct solution of nonlinear equations of a dynamic system by numerical integration on a computer; linearization of the system and the subsequent use of methods for studying linear dynamic systems. Schematically, using a dynamic element as an example, we show the idea as follows. The dynamic element equation has the form:

$$F(y, y', \dots, y^{(n)}, x, x', \dots, x^{(n)}) = 0.$$

This equation has a solution with respect to the highest derivative:

$$y^{(n)} = f(y, y', \dots, y^{(n)}, x, x', \dots, x^{(n)}),$$

followed by a transition to the system of differential equations with respect to the variables $y_1, \dots, y_n$:

$$\begin{gathered} y_1 = y, \\ \frac{dy_1}{dt} = y_2, \\ \vdots \\ \frac{dy_{n-1}}{dt} = y_n, \\ \frac{dy_n}{dt} = f(y, y', \dots, y^{(n)}, x, x', \dots, x^{(n)}). \end{gathered}$$

Next, it is necessary to obtain an analytical or numerical solution to an equation or a system of differential equations [19]. Non-linear Keynesian dynamic model can be represented as a first-order non-linear dynamic link:

$$\frac{dy}{dt} = f(y, I),$$

that is, GDP growth rate (y) is a function of GDP and investment. In the linear case,

$$f(y, I) = \underline{C} - (I - c)y + I.$$

It is obvious that

$$\frac{\partial f}{\partial y} < 0, \ \frac{\partial f}{\partial I} > 0,$$

therefore, the rate of GDP growth slows down with an increase in GDP, and it increases with an increase in investment.

Let us suppose that at $t = 0$ investments were equal to I_0 and the system was in some equilibrium state (y_E^0, I_0), the first component of which is determined from the equation of a nonlinear dynamic link:

$$f(y_E^0, I_0) = 0.$$

With an increase in investment from I_0 to $I = I_0 + \Delta I$, $\Delta I > 0$, the system satisfies the equation

$$\frac{dy}{dt} = f(y, I), \ y(0) = y_E^0.$$

Let us represent GDP as a sum of constant and variable parts:

$$y(t) = y_E^0 + \eta(t), \ \eta(t) > 0, \ \eta(0) = 0.$$

The variable part $\eta(t)$ satisfies the equation

$$\frac{dy}{dt} = f(y_E^0 + \eta, I_0 + \Delta I), \ \eta(0) = 0.$$

If the investment increment ΔI is relatively small, then with the evolutionary nature of the function $f(y, I)$, the variable part $\eta(t)$ is also relatively small, so the right side of the last equation can be expanded in the vicinity of the point (y_E^0, I_0) in a Taylor series, discarding terms of the second and higher orders:

$$\frac{dy}{dt} = \frac{\partial f}{\partial y}(y_E^0, I_0)\eta + \frac{\partial f}{\partial t}(y_E^0, I_0)\Delta I, \ \eta(0) = 0.$$

After transferring the term containing η to the left side and dividing both parts by $\frac{\partial f}{\partial y}(y_E^0, I_0)$, we get the equation of the inertial link:

$$T\frac{d\eta}{dt} + \eta = \alpha \Delta I, \ \eta(0) = 0,$$

where $\frac{1}{T} = -\eta \frac{\partial f}{\partial y}(y_E^0, I_0)$ is the generalized propensity to accumulate in the initial state;

$$\alpha = -\frac{\frac{\partial f}{\partial I}(y_E^0, I_0)}{\frac{\partial f}{\partial y}(y_E^0, I_0)} > 0.$$

From the last equation it follows that the variable part of GDP is equal to:

$$\eta(t) = \alpha \Delta I(1 - e^{\frac{1}{T}}),$$

and GDP is changed in general following the dependence:

$$y(t) = y_E^0 + \alpha \Delta I (1 - e^{\frac{1}{T}}),$$

in this case, the new equilibrium state of GDP is equal to [20]:

$$y_E = \lim_{t \to \infty} y(t) = y_E^0 + \alpha \Delta I = y_E^0 - \frac{\frac{\partial f}{\partial I}(y_E^0, I_0)}{\frac{\partial f}{\partial y}(y_E^0, I_0)} \Delta I.$$

Nonlinear multiply connected systems have seven types of stability and can have several equilibrium states. In such systems, the state of equilibrium can be either a fixed point or a closed trajectory (limit cycle). In both cases, particular solutions to differential equations are used.

Market cycles in the economy are described by a second-order linear dynamic model and are studied using a continuous analogue of the Samuelson-Hicks model or a continuous analogue of the non-linear Goodwin model. The Goodwin model [21] assumes that the capital intensity k, the population growth rate n, and the labor productivity γ remain constant:

$$k = \frac{K_t}{Y_t} = \text{const},$$

where K_t is the capital (fixed and current assets); $n = \dfrac{N_{t+1} - N_t}{N_t} = \text{const}$, where N_t is the population in the year t; $\gamma = \dfrac{y_{t+1} - y_t}{y_t} = \text{const},\ y_t = \dfrac{Y_t}{L_t}$ is the labor productivity, Y_t is the GDP, L_t is the number of employees.

The model has two endogenous variables λ_t and δ_t, where $\lambda_t = \dfrac{L_t}{N_t}$ is the share of employed people in the total population; $\delta_t = \dfrac{w_t L_t}{Y_t} = \dfrac{w_t}{y_t}$ is the share of the consumption fund in GDP, w_t is the annual wage rate.

The continuous analogue of the non-linear Goodwin model has the form:

$$\begin{cases} \dfrac{d\delta}{dt} = (a\lambda - a_0)\delta, \\ \dfrac{d\lambda}{dt} = (-b\delta + b_0)\lambda, \end{cases}$$

where $a = \dfrac{\alpha}{1+\gamma} > 0,\ a_0 = \dfrac{\alpha_0}{1+\gamma} > 0;\ b = \dfrac{1}{k(1+\gamma)(1+n)} > 0,\ b_0 = \dfrac{1 - k[\gamma - n(1+\gamma)]}{k(1+\gamma)(1+n)}$.

The control to a dynamic system is understood as a direct impact on the system in order to achieve a given result. Optimal control is understood as a choice from a set of alternative options for such control, which, according to a given criterion, is optimal. As an optimality criterion, a certain functional of the phase and control trajectories is chosen, which is subject to maximization (minimization).

The behavior of any nonlinear multiply connected system is described by the following equations of motion:

$$\frac{dy_i}{dt} = f_i(y, x, t),\ y_i(0) = y_i^0,\ i = 1, \ldots, n,$$

where y is the vector of phase coordinates that specifies the state of the system; x is the vector of external (input) setting and (or) disturbing influences on the system; y_i^0 are the initial values of phase variables.

If the disturbing actions are negligible, some of the setting actions become control actions, and others are given known functions of time, then we arrive at the following equations for the controlled dynamic system:

$$\frac{dy_i}{dt} = f_i(y, u, t),\ y_i(0) = y_i^0,\ i = 1, \ldots, n,$$

where u is the vector of control parameters, $u \in U$; U is the area of acceptable values of control parameters.

The control trajectory (control) $u(t)$ is called admissible if it is piecewise continuous, continuous at the discontinuity points on the left:

$$u(\tau) = u(\tau - 0) = \lim_{i \to \tau} u(t),$$

and moreover, for any $t < \tau$ $u(t) \in U$. If the control law is given, i.e. an admissible control trajectory $u(t)$ is defined, then the equations for the phase variables take the form:

$$\frac{dy_i}{dt} = f_i(y, u(t), t),\ y_i(0) = y_i^0,\ i = 1, \ldots, n,$$

thus, for any initial conditions $y(0) = y^0$, the solution is uniquely determined. As an optimality criterion, a certain functional of the phase and control trajectories is chosen, which is subject to maximization (minimization) [22]. The necessary conditions for solving such a problem are given by the Pontryagin maximum principle [23].

The Pontryagin maximum principle is applied to a general control problem of the form

$$\max_{u(t) \in U} \int_0^T f_0(y, u, t)dt + F(y^T, T),$$

$$\frac{dy}{dt}t = f(y, u, t),\ y(0) = y^0,$$

where $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is the column-vector of phase variables that determine the state of the dynamic system; $f(y, u, t) = \begin{pmatrix} f_1(y, u, t) \\ \vdots \\ f_n(y, u, t) \end{pmatrix}$ is the column-vector of the right parts of the equations of the system; y^0, y^T are the initial and final values of the state vector; $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ is the column-vector of control parameters; U is the area of possible values of control parameters; $f_1(y, u, t)$ is the integrand of the control criterion.

The functions $f_i(y, u, t)$, $F(y^T, T)$ are continuous and differentiable with respect to each argument. If the equation $u(t)$ is defined, then the trajectory of the system $y(t)$ is uniquely defined for the given initial condition $y(0) = y^0$. The search for the trajectory of the system corresponding to the optimal control is reduced to finding the saddle point of the Lagrange function in a nonlinear programming problem.

We draw the following conclusion: economic phenomena and processes are characterized by non-linearity. However, the use of the mathematical method of system linearization makes it possible to apply methods for studying linear dynamic systems for the analysis of economic entities.

5. Small-sector non-linear dynamic models of macroeconomics.

Non-linear small-sector models are used to study long-term trends, growth factors, and assess the consequences of options for macroeconomic decisions. The base model is the one-sector Solow model [24]. In this model, the economic system is considered as a single unstructured whole that produces one universal product. In this case, the product can be both consumed and invested. The model in its most aggregated form reflects the process of reproduction and allows to analyze the relationship between consumption and accumulation in general terms.

The state of the economy is given by five endogenous variables: X is GDP; C is a non-productive consumption fund; I are investments; L is a number of employees; K are basic production assets.

The model uses three exogenous indicators: ν is the annual growth rate for the number of employees; μ is the share of fixed production assets retired during the year; ρ is the rate of accumulation (share of gross investment in GDP). Exogenous indicators are within the following limits: $-1 < \nu < 1,\ 0 < \mu < 1,\ 0 < \rho < 1$. It is assumed that endogenous variables change over time, while exogenous indicators are constant.

The Solow model in absolute terms is

$$L = L_0 e^{\nu t};\ \frac{dK}{dt} = -\mu K + \rho X;\ K(0) = K_0;$$

$$X = F(K, L);\ I = \rho X;\ C = (1-\rho)X.$$

This model takes into account two aggregated products (means of production and commodities) and two sectors. The first sector produces means of production, the second one produces consumer goods.

The Solow model solves the problem known as the "Golden Rule of Accumulation" [25]. Its essence boils down to the fact that in a stationary mode, with a proper choice of the rate of accumulation, in a relatively short period of time after the start of the transition process, it is possible to maximize per capita consumption. Indeed,

$$c^E(p) = (1-p)A(k^E)^\alpha = (1-p)A[\frac{\rho A}{\lambda}]^{\frac{\alpha}{1-\alpha}} = B[g(\rho)]^{\frac{1}{1-\alpha}},$$

where $B = [\frac{A}{\lambda^\alpha}]^{\frac{1}{1-\alpha}},\ g(\rho) = \rho^\alpha(1-\rho)^{1-\alpha}$. Thus, per capita consumption c is entirely determined by the function $g(\rho)$.

We have

$$\frac{dg}{d\rho} = (\frac{\rho}{1-\rho})^\alpha \frac{\alpha-\rho}{\rho},$$

therefore $\frac{dc^E}{d\rho} > 0$ for $\rho < \alpha$, $\frac{dc^E}{d\rho} < 0$ for $\rho > \alpha$. Thus, the highest average per capita consumption is achieved at $\rho^* = \alpha$, i.e. the rate of accumulation should be equal to the elasticity of output for funds.

With $\rho = \text{const}$ and current consumption per employee $c(t) = C(t)/L(t)$, the Solow model is transformed into a one-sector model of optimal economic growth [26]:

$$\frac{dk}{dt} = f(k) - (\mu + \nu)k - c,\ k(0) = k_0,$$

since the quantity $\rho f(k)$ in the Solow model is replaced by $f(k) - c(t)$. The last equation is the main equation of the controlled system.

Specific consumption $c(t)$ is considered as a control parameter. As it is customary in optimal control theory, its admissible trajectory can be any piecewise continuous trajectory that satisfies the boundary value condition:

$$0 < \underline{c} \leq c(t) \leq f(k(t)),$$

where $\underline{c}$ is the maximum permissible lower limit of specific consumption.

The problem of the governing body of the economic system is to choose the value of current consumption in such a way that, over a long period of time, the discounted utility from consumption is maximum:

$$\int\limits_0^\infty e^{-\delta t} u(c(t))dt \to \max,$$

where δ is the discount rate by which future utilities are reduced to the present (assuming that immediate consumption is more important than distant consumption); $u(c)$ is the consumption utility function.

The process of reproduction is reflected in more details by a three-sector model of the economy, in which there are three aggregated products (objects of labor, means of labor and consumer goods), and each of the three sectors produces its own product, namely material sector produces objects of labor, capital-creating – means of labor, and consumer – consumer goods [27].

When constructing a three-sector model of the economy, we assume the following.

1. The technological structure is considered constant and is set using linearly homogeneous neoclassical production functions

$$X_i = F_i(K_i, L_i),$$

where X_i, K_i, L_i are the output, fixed production assets and the number of people employed in the i-th sector.

2. The total number of employed L in the manufacturing sector changes with the constant growth rate ν.

3. There is no investment lag.

4. The depreciation coefficients of fixed production assets μ_i and the direct material costs α_i of the sectors are constant.

5. The economy is closed, i.e. foreign trade is not considered.

6. The time t changes continuously.

Assumption (2) in discrete time has the following form (t is the number of the year):

$$\frac{L(t+1) - L(t)}{L(T)} = \nu$$

and upon transition to continuous time, it takes the form of a differential equation

$$\frac{dL}{dt} = \nu L,\ L(0) = L^0,$$

which has a solution

$$L = L^0 e^{\nu t}.$$

From assumptions (3), (4) it follows that the change over the year of the fixed production assets of the i-th sector consists of two parts: the depreciation $(\mu_i K_i)$ and a growth due to the gross capital investments $(+I_i)$, i.e.

$$K_i(t+1) - K_i(t) = -\mu K_i(t) + I_i(t),\ i = 0, 1, 2,$$

or in continuous time

$$K_i(t+\Delta t) - K_i(t) = -[\mu K_i(t) + I_i(t)]\Delta t,$$

with $\Delta t \to 0$ we obtain differential equations for the main production assets of the sectors

$$\frac{dK_i}{dt} = -\mu K_i + I_i,\ K_i(0) = K_i^0,\ i = 0, 1, 2.$$

Thus, under the assumptions made, the three-sector model of the economy (with the signs of time omitted) in absolute terms takes the form: $L = L(O)e^{\nu t}$ is the number of employees; $L_0 + L_1 + L_2 = L$ is the distribution of employed by sectors; $\frac{dK_i}{dt} = -\mu K_i + I_i,\ K_i(0) = K_i^0,\ i = 0, 1, 2$ is the dynamics of funds by sectors; $X_i = F_i(K_i, L_i),\ i = 0, 1, 2$ is the output by sectors; $X_1 = I_0 + I_1 + I_2$ is the distribution of products of the fund-creating sector; $X_0 = a_0 X_0 + a_1 X_1 + a_2 X_2$ is the distribution of products of the material sector.

With the help of a three-sector model, we identify conditions under which the economy falls into stagnation or balanced economic growth. It is proved that in a stationary state a three-sector economy has a technological optimum. Therefore, any change in the stationary state of the economy as a result of an external influence or a control decision can be assessed as positive if there is a movement towards the optimum point, and negative otherwise.

We draw the following conclusion: based on the basic one-sector Solow model, an arsenal of models was developed and recommended for a small-sector study of the state of the economy and determining its technological optimum.

Table 2

Mathematical modelling of microeconomics

Method	**Model**	**References**
6. Consumer behavior models	Consumer preferences and utility function	[28]
	Consumer behavior model	[29]
7. Producer behavior models	Firm model	[30]
	Duopoly model	[31]
8. Models of interaction between consumers and producers	Equilibrium price model	[32]
	Walrasian model	[33]

6. The household (consumer) is an important concept in microeconomics. In the study of consumer behavior, the main problem is to establish the magnitude of his demand for purchased goods and services at given prices and his income.

A consumer's decision to buy a certain set of goods can be mathematically represented as a choice of a specific point in the space of goods. Let n be a finite number of goods under consideration; $x = (x_i, \ldots, x_n)^i$ be a column-vector of volumes of goods purchased by the consumer for a certain period at given prices and income for the same period. The space of goods is the set of possible sets of goods x with non-negative coordinates:

$$C = \{x : x \geq 0\}.$$

In consumer choice theory [28], it is assumed that each consumer initially has his own preferences on some subset of the product space $X \subset \{x : x \geq 0\}$. This means that for every pair $x \ni X,\ y \ni Y$ one of three relations takes place:

$x \succ y$, i.e. the set x is preferred over y;

$x \prec y$, i.e. the set x is less preferred than y;

$x \sim y$, i.e. for the consumer, both sets have the same degree of preference.

Preference relations have the following properties:

if $x \succ y$, $y \succ z$, then $x \succ z$ (transitivity);

if $x > y$, then $x \succ y$ (unsaturation: a larger set is always preferable to a smaller one).

The preference relations of each consumer can be represented in the form of a preference indicator, i.e. the utility function $u(x)$ such that $x \succ y$ implies $u(x) > u(y)$ and $x \sim y$ implies $u(x) = u(y)$. For each consumer, such a representation is multivariate. The introduction of a utility function makes it possible to replace preference relations with the usual relations between numbers: greater than, less than, equal to.

In the model of consumer behavior [29], it is assumed that the consumer always seeks to maximize his utility and is constrained only by limited income:

$$\max_{x \in \delta \cap X} u(x) = \max_{px \geq M} u(x).$$

This conditional extremum problem is reduced to finding the unconditional extremum of the Lagrange function:

$$L(x) = u(x) + \lambda(M - px).$$

Necessary conditions for a local extremum are as follows:

$$\sum_{j=1}^{n} p_j x_j^* = M,$$

$$\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i}(x_i^*) - \lambda^* p_i = 0,\ i = 1, \ldots, n.$$

These conditions really determine the maximum point, since the matrix U is negative definite.

We draw the following conclusion: the developed mathematical arsenal allows to determine the behavior of the consumer – his preferences and usefulness in the face of budget constraints – not on a qualitative, but on a quantitative level.

7. When studying the behavior of a manufacturer in the firm model [30], it is assumed that a manufacturing firm produces one type of product or several types, but with a constant structure: X is the number of units of one type of product or the number of multi-product units. Each of the three aggregated types of resources (labor L, funds K, materials M) has a certain number of varieties. The technology of a firm is determined by its production function, which expresses the relationship between resource inputs and output:

$$X = F(x),$$

where $x = (x_1, \ldots, x_n)$ is a column-vector of possible costs for various types of resources. It is assumed that $F(x)$ is twice continuously differentiable and neoclassical. Moreover, the matrix of its second derivatives is negative definite. If the price of a unit of production is equal to p, and the price of a unit of a resource of the j-th type is $w_j,\ j = 1, \ldots, n$, then each cost vector x corresponds to the profit

$$\Pi(x) = pF(x) - wx$$

where $w = (w_1, w_2, \ldots, w_n)$ is a row-vector of resource prices.

In the presence of a natural restriction on non-negativity of the sizes of resources involved in production, the problem of maximizing profit takes the form:

$$\max_{(x \geq 0)}[pF(x) - wx].$$

This is a non-linear programming problem with n non-negativity conditions $x \geq 0$. The necessary conditions for its solution are the Kuhn – Tucker conditions:

$$\frac{\partial \Pi}{\partial x} = p\frac{\partial F}{\partial x} - w \leq 0,$$

$$\frac{\partial \Pi}{\partial x}x = (p\frac{\partial F}{\partial x} - w)x = 0,$$

$$x \geq 0.$$

If all types of resources are used in the optimal solution, i.e. $x^* > 0$, then the solution to the problem takes the form:

$$p\frac{\partial F(x^*)}{\partial x} = w \ or \ p\frac{\partial F(x^*)}{\partial x_j} = w_j,\ j = 1, \ldots, n,$$

that is the optimal point, the value of the marginal product of a given resource must be equal to its price.

In the most general case of the duopoly model [31], two competitors produce one type of product in accordance with their production function

$$X_i = F_i(x^i),\ i = 1, 2.$$

The price of products depends on both issues:

$$p = p(X_1, X_2),$$

and as output increases, the price falls:

$$\frac{\partial p}{\partial X_1} < 0,\ \frac{\partial p}{\partial X_2} < 0.$$

The resource price also depends on the volume of its purchases $x_j^1,\ x_j^2$ by the first and second firms:

$$w_j = w_j(x_j^1, x_j^2),\ j = 1, \ldots, n,$$

where prices rise as demand increases:

$$\frac{\partial w_j}{\partial x_j^1} < 0,\ \frac{\partial w_j}{\partial x_j^2} < 0.$$

Every firm seeks to maximize its profits. For example, the first firm should act as follows:

$$\max_{(X_1, x_1^1, \ldots, x_n^1)} [p(X_1, X_2)X_1 - \sum_{j=1}^{n} w_i(x_j^1, x_j^2)x_j^1]$$

under the condition $X_1 = F_1(x_1^1, \ldots, x_n^1)$.

The Lagrange function for this problem has the form:

$$L(X_1, x^1, \lambda) = p(X_1, X_2)X_1 - \sum_{j=1}^{n} w_i(x_j^1, x_j^2)x_j^1 + \lambda(F_1(x_1^1, \ldots, x_n^1) - X_1),$$

$$\frac{\partial L}{\partial X_1} = p(X_1, X_2) + X_1 \frac{\partial p}{\partial X_1} + X_1 \frac{\partial p}{\partial X_2} \frac{\partial X_2}{\partial X_1} - \lambda = 0,$$

$$\frac{\partial L}{\partial x_j^{(1)}} = -w_j(x_j^1, x_j^2) - x_j^1 \frac{\partial w_j}{\partial x_j^1} - x_j^1 \frac{\partial w_j}{\partial x_j^2} \frac{\partial x_j^2}{\partial x_j^1} + \lambda \frac{\partial F_1}{\partial x_j^1} = 0, \ j = 1, \ldots, n,$$

$$\frac{\partial L}{\partial \lambda} = F_1(x_1^1, \ldots, x_n^1) - X_1 = 0.$$

Eliminating λ, we get $(n+1)$ equations for determining the strategy $X_1, x_1^1, \ldots, x_n^1$ of the first firm:

$$\left[p(X_1, X_2) + \left(X_1 \frac{\partial p}{\partial X_1} + X_1 \frac{\partial p}{\partial X_2} \frac{\partial X_2}{\partial X_1}\right)\right] \frac{\partial F_1}{\partial x_j^1} = w_j + x_j^{(1)} \left(\frac{\partial w_j}{\partial x_j^1} + \frac{\partial w_j}{\partial x_j^2} \frac{\partial x_j^2}{\partial x_j^1}\right), \ j = 1, \ldots, n,$$

$$X_1 = F_1(x_1^1, \ldots, x_n^1).$$

The solution to these equations depends on $\dfrac{\partial X_2}{\partial X_1}$ and $\dfrac{\partial x_j^2}{\partial x_j^1}$, $j = 1, \ldots, n$. The latter equations represent the expected response of the second firm to the strategy $X_1, x_1^1, \ldots, x_n^1$ of the first firm. Under different assumptions about this response, different solutions to the competitors' problem are obtained in the duopoly model. We draw the following conclusion: based on the production functions, the above models allow to determine the optimal strategy of the company in a competitive environment.

8. Models for establishing an equilibrium price in the processes of interaction between consumers and producers are based on the assumption that price changes depend on the difference between supply and demand: if demand is higher than supply, then the price increases, otherwise it decreases.

The most well-known model for establishing an equilibrium price in the market for one product is the "cobweb" one [32]. In this model, demand is characterized by the decreasing aggregate demand function $\Phi(p)$, while supply is characterized by the increasing aggregate supply function $\psi(p)$. These functions are defined and continuous for all $p > 0$. Moreover,

$$\lim_{p \to 0} \Phi(p) = \infty, \ \lim_{p \to \infty} \Phi(p) = 0,$$

$$\lim_{p \to 0} \psi(p) = 0, \ \lim_{p \to \infty} \psi(p) = \infty.$$

The state of equilibrium is characterized by the equality of supply and demand:

$$\Phi(p) = \psi(p),$$

notably, by virtue of the assumptions made, the last equation has the unique solution p^E, so that the equilibrium state

$$\Phi(p^E) = \psi(p^E) = x^E$$

is unique.

The "cobweb" model makes it possible to implement the process of iterative approximation to the equilibrium price. Let us assume that at the initial moment of time the price p_0 is set, while the demand turned out to be less than the supply:

$$\Phi(p_0) < \psi(p_0),$$

then in the model we lower the price to a level at which demand is equal to supply at the initial price:

$$\Phi(p_1) = \psi(p_0).$$

At the new price p_1, demand exceeds supply:

$$\Phi(p_1) > \psi(p_1),$$

therefore, we raise the price to the level p_2, at which

$$\Phi(p_2) = \psi(p_1),$$

and so on. Thus, the process described by the recurrent relation $\Phi(p_1) = \psi(p_1)$, $i = 1, 2, \ldots$, converges.

The Walrasian model [33] considers an economy with I consumers ($i = 1, \ldots, I$), m producers ($k = 1, \ldots, m$) and n types of goods ($j = 1, \ldots, n$). Let $p = (p_1, , p_n)$ be a row-vector of prices and $x = (x_1, \ldots, x_n)$ be a column-vector of goods.

Each consumer has the income $K(p)$ and his own preference field for goods, which can be specified as the utility function $u(x)$. If we denote the set of possible sets of goods available to the consumer at prices p by $X(p) = \{x^* : x \in X,\ px \leq K(p)\}$, X is the domain of definition of $u(x)$, then the consumer demand function is given by

$$\Phi(p) = \begin{cases} x^* : x \in X(p),\ u(x^*) = \max\limits_{x \in X(p)} u(p) \\ 0, \qquad\qquad\qquad u(x^*) \neq \max\limits_{x \in X(p)} u(p) \end{cases},$$

i.e. the demand function is the set of available goods, each of which maximizes consumer utility at the given price p. It is assumed that the income of each consumer consists of two parts: the income pb_i from the sale of the initial stock of goods b_i and the income $I_i p$ as a result of the consumer's participation in production, i.e. $K_i(p) = pb_i + I_i p$.

Each manufacturer (firm) is set by its technological capabilities. Let us denote the input-output column-vector of the k-th producer by $c = (y_{k1}, \ldots, y_{kn})$: the positive components of this vector define the firm's output, the negative components define the costs. Therefore, the dot product py_k represents the profit of the firm. The technological capabilities of a firm are defined as the set of admissible input-output vectors Y_k. This set is called the production possibilities set.

The distribution of production is carried out by choosing the input-output vector y_k from the technological set of production possibilities Y_k for each producer $k =$

$1, \ldots, m$. The sum $Y = \sum_{k=1}^{m} y_k$ represents the overall production process. Distribution of consumption is carried out by choosing a consumption menu $x_i \in X_i, i = 1, \ldots, l$ by each consumer. The sum $x = \sum_{i=1}^{l} x_i$ is a vector of aggregate demand, some of components of which may be negative if they represent supply (for example, labor).

The joint distribution of production and consumption is understood as such a set of consumption vectors and the input-output vectors $(x_1, \ldots, x_i, \ldots, x_l, y_1, \ldots, y_k, \ldots, y_m)$, $x_i \in X_i$, $y_k \in Y_k$, for which the aggregate demand matches the total offer:

$$x = \sum_{i=1}^{l} x_i = b + \sum_{k=1}^{m} y_k = b + y.$$

The set $(x_1^*, \ldots, x_i^*, \ldots, x_l^*, y_1^*, \ldots, y_k^*, \ldots, y_m^*, p^*)$ defines a competitive equilibrium in the Walrasian model if

$$x_i^* \in \Phi_i(p^*),\ i = 1, \ldots, l,\ y_k^* \in \psi_k(p^*),\ k = 1, \ldots, m,$$

$$\sum_{k=1}^{m} y_k^* + b \geq \sum_{i=1}^{l} x_i^*$$

$$p(\sum_{k=1}^{m} y_k^* + b) = p^* \sum_{i=1}^{l} x_i^*$$

In this case, p^* is called the vector of competitive prices, and the last two equations are called the Walras's law. We draw the following conclusion: the above mathematical models are of great practical importance in the processes of establishing an equilibrium price in the interactions of consumers and producers.

Despite the fact that most of the considered mathematical models of macroeconomics and microeconomics were developed relatively long ago, at the beginning and in the middle of the last century, they have not lost their relevance to the present day and are widely demanded by practitioners in the study of economic phenomena and processes. This is evidenced by a far from complete list of references cited.

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МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ЭКОНОМИКИ

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В статье представлен обзор основных методов моделирования экономики, используемых в научных исследованиях за последние двадцать лет. Обзор не претендует на охват всех направлений, методов и моделей, используемых в научных исследованиях в области экономики. Это невозможно сделать в рамках одной статьи. Рассмотрено математическое моделирование только двух разделов экономической теории: макроэкономики и микроэкономики. При этом отсутствует литературный обзор методов и моделей исследования в разделе микроэкономики, которые имеются в инструментарии научного исследования, но уже описаны в разделе макроэкономики. Надеемся, что данный обзор будет полезен ученым, занимающимся опосредованным изучением экономических явлений и процессов.

Ключевые слова: моделирование; модель; макроэкономика; микроэкономика; источники; обзор.

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