ПРОГРАММИРОВАНИЕ

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NOTE ON EXACT FACTORIZATION ALGORITHM FOR MATRIX POLYNOMIALS

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There are two major obstacles for a wide utilisation of the Wiener–Hopf factorization technique for matrix functions used to solve vectorial Riemann boundary problems. The first one reflects the absence of a general explicit factorization method in the matrix case, even though there are some explicit (constructive) factorizations available for specific classes of matrix functions. The second obstacle follows from the fact that the factorization of a matrix function is, generally speaking, not stable operation with respect to a small perturbation of the original function. As a result of the latter, a realisation of any constructive algorithm, even if it exists for the given matrix function, cannot be performed in practice. Moreover, developing explicit methods, authors do not often analyze its numerical implementation, implicitly assuming that all steps of the proposed constructive algorithm can be carried out exactly. In the proposed work, we continue studying a relation between the explicit and exact solutions of the factorization problem in the class of matrix polynomials. The main goal is to obtain an algorithm for the exact evaluation of the so-called indices and essential polynomials of a finite sequence of matrices. This is the cornerstone of the problem of exact factorization of matrix polynomials.

Keywords: Wiener–Hopf factorization; toeplitz matrices; essential polynomials of sequence.

Introduction

Let $A(t)$ be a matrix function from the matrix Wiener algebra $W^{p\times p}(\mathbb{T})$ that is invertible on the unit circle $\mathbb{T}$. The representation

$$A(t) = A_+(t)D(t)A_-(t), \quad t \in \mathbb{T}, \tag{1}$$

is called *a left Wiener–Hopf factorization* of $A(t)$. Here $A_\pm(t)$ belong to the group $GW^{p\times p}_\pm(\mathbb{T})$ of invertible elements of the subalgebra $W^{p\times p}_\pm(\mathbb{T})$, the middle factor $D(t)$ is the diagonal matrix $D(t) = \mathrm{diag}\left[t^{\lambda_1}, \ldots, t^{\lambda_p}\right]$, where integers $\lambda_1 \geq \ldots \geq \lambda_p$ are *the left partial indices* of $A(t)$. The relation $\lambda_1 + \ldots + \lambda_p = \varkappa = \mathrm{ind}_{\mathbb{T}} \det A(z)$ is valid. A similar representation in which the factors $A_\pm$ are rearranged is called *the right Wiener–Hopf factorization.*

Mathematical modelling of wave diffraction, problems of dynamic elasticity and fracture mechanics, and geophysical problems are often reduced to the Wiener–Hopf factorization problem for matrix functions [1–4]. The factorization of matrix functions is also a powerful tool itself used in various areas of mathematics [5–7, 9].

Unfortunately, for the matrix case, there is no constrictive solution of the factorization problem in a general setting and it is very important to find cases when the problem can be solved effectively or explicitly. By the *explicit* (or *constructive*) solution of the factorization problem we understand a clearly defined algorithmic procedure that should definitely terminate after a finite number of steps. There are not that many classes of matrix functions for which an explicit solution to factorization problem has been found.

The most important of them are classes of matrix polynomials [10, 11] and meromorphic matrix functions [12]. A detailed review of constructive methods for the factorization problem is presented in the works [13–15].

In addition to the aforementioned lack of availability of explicit solution to the factorization problem, in the general case, there is another obstacle to use the technique. This is possible instability of the factorization problem. Even if an explicit method for solving the particular factorization problem exists, each step of the respective algorithm can be executed exactly or approximately (numerically).

We say that the problem can be solved *exactly* if **(i)** the input data belonging to the Gaussian field $\mathbb{Q}(i)$ of complex rational numbers, and **(ii)** all steps of the explicit algorithm can be perform in the *exact arithmetic*. The instability of the problem leads to the fact that the explicit algorithm cannot be implemented numerically. As a rule, researchers developing a particular explicit factorization method usually ignore this issue. In fact, they implicitly assume that all steps of the proposed explicit algorithm can be carried out exactly that, unfortunately, is not always possible.

For the first time, the need to accurate study the way of numerical implementation of the explicit algorithm was highlighted in [16]. This has been done for matrix polynomials, where existence of an explicit solution of the factorization problem was proved in [11]. In [16], based on this work, a criterion for the *exact* factorizability of a matrix polynomial was obtained, and an *exact* algorithm for a solution of the factorization problem was developed. This algorithm was also implemented as the package ExactMPF in Maple. Thus, if the condition satisfies, the problem of an instability does not arise.

The package makes it easy to carry out numerical experiments with the Wiener–Hopf factorization for matrix polynomials. It can be used to construct an approximate canonical factorization with quaranteed accuracy for strictly nonsingular 2×2 matrix functions and to the integration of a discrete analog of the nonlinear Schrödinger equation by Inverse Scattering Transform. We hope that the application of the package will not be exhausted by these examples.

This paper is complimentary to [16], where the length was limited by the publisher rules. As a result, some crucial technical results have been omitted there. In particular, the algorithm for constructing essential polynomials was not described. In this work we fill this gap.

1. Explicit Solution of the Factorization Problems for Matrix Polynomials

In this section, we present an explicit algorithm for the factorization of an arbitrary matrix polynomial. Our presentation is based on the results from [11, 12, 16].

Supposed that the matrix polynomial $a(z) = \sum\limits_{k=0}^{N} a_k z^k$, $a_k \in \mathbb{C}^{p\times p}$, is invertible on the unit circle $\mathbb{T}$. We will write its left and right Wiener–Hopf factorizations in the form

$$a(t) = l_+(t) d_L(t) l_-(t), \quad a(t) = r_-(t) d_R(t) r_+(t), \ t \in \mathbb{T}. \tag{2}$$

Here $d_L(t) = \mathrm{diag}[t^{\lambda_1}, ..., t^{\lambda_p}]$, and $\lambda_1 \geq \cdots \geq \lambda_p$; $d_R(t) = \mathrm{diag}\,[t^{\rho_1}, \ldots, t^{\rho_p}]$, where $\rho_1 \leq \ldots \leq \rho_p$. Note that left λ_j and right ρ_j indices are usually different sets of integers and constructions of the right and left factorizations are usually considered as two separate problems. For explicit construction of these factorizations we will use the method proposed in the work [11]. The method requires simultaneous considerations of the both factorizations.

Let $\Delta(z) = \det a(z)$ and $\Delta(z) = \Delta_-(z)z^{\varkappa}\Delta_+(z)$, $\Delta_-(\infty) = 1$, be the Wiener–Hopf factorization of $\Delta(z)$. The factorization is unique with the additional condition at infinity for the polynomial $\Delta_-(z)$. In the sequel, we use, in fact, only one of the factors, namely, $\Delta_-(z) = 1 + \Delta_1^- z^{-1} + \cdots + \Delta_{\varkappa}^- z^{-\varkappa}$, $\varkappa = \mathrm{ind}_{\mathbb{T}} \det a(z)$.

We expand the rational matrix function $\Delta_-^{-1}(z)a(z)$ in the Laurent series at infinity: $\Delta_-^{-1}(z)a(z) = \sum_{j=-\infty}^{N} c_j z^j$. The coefficients $c_j \in \mathbb{C}^{p\times p}$ are computed recurrently in terms of matrix coefficients a_j of the original matrix polynomial, $a(z)$, and the coefficients Δ_j^- of the scalar polynomial, $\Delta_-(z)$, (see [16, Eq.(2.5)]).

To construct the factorizations of $a(z)$ we only need a finite number of the coefficients, c_k, for $k = -\varkappa, \ldots, 0, \ldots, \varkappa$. Denote $c_{-\varkappa}^{\varkappa} := \{c_{-\varkappa}, \ldots, c_0, \ldots, c_{\varkappa}\}$. The main tools for computations of the partial indices and factors in the factorizations of matrix functions are the so-called indices and essential polynomials of the sequence $c_{-\varkappa}^{\varkappa}$ (see [11, 17]). Let us define these notions.

Form a finite family of the block Toeplitz matrices of finite sizes:

$$T_k = \|c_{i-j}\|_{\substack{i=k,k+1,\ldots,\varkappa \\ j=0,1,\ldots,\varkappa+k}}, \quad -\varkappa \le k \le \varkappa, \tag{3}$$

and study the structure of the right $\ker_R T_k = \{R \in \mathbb{C}^{(k+\varkappa+1)\times 1} | T_k R = 0\}$ and left $\ker_L T_k = \{L \in \mathbb{C}^{1\times(\varkappa-k+1)} | L T_k = 0\}$ kernels of T_k. Further it is more convenient to deal not with vectors $R = (r_0, r_1, \ldots, r_{k+\varkappa})^T \in \ker_R T_k$, $r_j \in \mathbb{C}^{p\times 1}$, but with their generating column-valued polynomials $R(z) = r_0 + r_1 z + \cdots + r_{k+\varkappa} z^{k+\varkappa}$. We will use the spaces $\mathcal{N}_k$ of the generating polynomials instead of the spaces $\ker T_k$.

By $\mathcal{N}_k^R$, $-\varkappa \le k \le \varkappa$, we denote the space of generating vector polynomials for vectors in $\ker_R T_k$. Put $\mathcal{N}_{-\varkappa-1}^R := \{0\}$ and let $\mathcal{N}_{\varkappa+1}^R$ be $(2\varkappa+2)p$–dimensional space of all column-valued polynomials whose degrees are not greater than $2\varkappa+1$.

Repeating the same line of reasoning, we can define the space $\mathcal{N}_k^L$, $-\varkappa \le k \le \varkappa$, of the row-valued generating polynomials in z^{-1} for the rows from $\ker_L T_k$.

By d_k^R, we denote a dimension of the right kernel $\mathcal{N}_k^R$ and introduce the following integers: $\Delta_k^R = d_k^R - d_{k-1}^R$ for $-\varkappa \le k \le \varkappa+1$. A sequence $c_{-\varkappa}^{\varkappa}$ is called *regular* if $\Delta_{-\varkappa}^R = 0$ and $\Delta_{\varkappa+1}^R = 2p$.

For a regular sequence, we have (see [11, 17])

$$0 = \Delta_{-\varkappa}^R \le \Delta_{-\varkappa+1}^R \le \cdots \le \Delta_{\varkappa}^R \le \Delta_{\varkappa+1}^R = 2p.$$

Since a monotone integer sequence is piecewise constant, then there are $2p$ integers $\mu_1 \le \cdots \le \mu_{2p}$ such that

$$\begin{array}{ccccccc} \Delta_{-\varkappa}^R & = & \cdots & = & \Delta_{\mu_1}^R & = & 0, \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \Delta_{\mu_i+1}^R & = & \cdots & = & \Delta_{\mu_{i+1}}^R & = & i, \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \Delta_{\mu_{2p}+1}^R & = & \cdots & = & \Delta_{\varkappa+1}^R & = & 2p. \end{array} \tag{4}$$

The absence of the j–th row here means that $\mu_{j+1} = \mu_j$.

Definition 1. *The integers $\mu_1, \ldots, \mu_{2p}$ defined by the relations (4) are the indices of the sequence $c_{-\varkappa}^{\varkappa}$.*

Similarly, we can consider the sequence of the left kernel $\mathcal{N}_k^L$ that will lead, however, to the same indices.

Furthermore, we define the right essential polynomials of the sequence $c_{-\varkappa}^{\varkappa}$. Note that $\mathcal{N}_k^R$ and $z\mathcal{N}_k^R$ are subspaces of $\mathcal{N}_{k+1}^R$ as it follows from the definition of the spaces $\mathcal{N}_k^R$. The dimension, h_{k+1}^R, of the complement $\mathcal{H}_{k+1}^R$ of $\mathcal{N}_k^R + z\mathcal{N}_k^R$ in $\mathcal{N}_{k+1}^R$ is equal to $\Delta_{k+1}^R - \Delta_k^R$.

Then, Eqs. (4) imply that $h^R_{k+1} \neq 0$ if and only if $k = \mu_j$, $j = 1, \dots, 2p$. Moreover, in this case, h^R_{k+1} is equal to the multiplicity, $\varkappa_j$, of the index μ_j. Therefore, for $k \neq \mu_j$ we have

$$\mathcal{N}^R_{k+1} = \mathcal{N}^R_k + z\mathcal{N}^R_k,$$

and for $k = \mu_j$

$$\mathcal{N}^R_{k+1} = \left(\mathcal{N}^R_k + z\mathcal{N}^R_k\right) \oplus \mathcal{H}^R_{k+1}.$$

Definition 2. *Any polynomials $R_j(z), \dots, R_{j+\varkappa_j-1}(z)$ forming a basis for a complement $\mathcal{H}^R_{\mu_j+1}$ are called right essential polynomials of the sequence $c^{\varkappa}_{-\varkappa}$ corresponding to the index μ_j.*

As a result, we have defined $2p$ indices $\mu_1, \dots, \mu_{2p}$ and $2p$ right essential polynomials $R_1(z), \dots, R_{2p}(z)$ for any regular sequence $c^{\varkappa}_{-\varkappa}$. Similarly, we can define the left essential polynomials $L_1(z), \dots, L_{2p}(z)$ of the sequence $c^{\varkappa}_{-\varkappa}$.

In factorization problems, there are natural candidates for the role of indices and essential polynomials. To check that this is indeed the case, the following essentiality criterion can be used ([17, Theorem 4.1], see also [11, p. 258],):

Theorem 1. *The integers $\mu_1, \dots, \mu_{2p}$ are the indices and $R_1(z), \dots, R_{2p}(z)$ are right essential polynomials of the regular sequence $c^{\varkappa}_{-\varkappa}$ if and only if the matrix*

$$\Lambda_R = \begin{pmatrix} \widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\} & \cdots & \widetilde{\sigma}_R\{z^{-\varkappa-1}R_{2p}(z)\} \\ R_1(0) & \cdots & R_{2p}(0) \end{pmatrix}$$

is invertible. Here $\widetilde{\sigma}_R\{z^{-\varkappa-1}R_j(z)\} = \sum_{m=1}^{\varkappa+\mu_j+1} c_{\varkappa+1-m} r^{(j)}_m$.

By Theorem 3.1 from [11], the sequence $c^{\varkappa}_{-\varkappa}$ is regular and there exist respective essential polynomials $R_1(z), \dots, R_{2p}(z)$; $L_1(z), \dots, L_{2p}(z)$ such that

(i) the constant terms of the polynomials $R_1(z), \dots, R_p(z)$ are equal to zero,

(ii) the leading terms of the polynomials $L_{p+1}(z), \dots, L_{2p}(z)$ are equal to zero.

Definition 3. *The essential polynomials $R_1(z), \dots, R_{2p}(z)$; $L_1(z), \dots, L_{2p}(z)$ satisfying the conditions (i), (ii) are called the factorization essential polynomials of the sequence.*

Now we can formulate a final result on the explicit Wiener–Hopf factorization of a matrix polynomial $a(z)$.

Theorem 2. [11, Theorem 3.2] *Let $\mu_1, \dots, \mu_{2p}$ be the indices and $R_1(z), \dots, R_{2p}(z)$ $\big(L_1(z), \dots, L_{2p}(z)\big)$ are the right (left) factorization essential polynomials of the regular sequence $c^{\varkappa}_{-\varkappa}$. Let us introduce the $p \times p$ matrix functions*

$$\mathcal{R}_1(z) = \left(R_1(z) \ \dots \ R_p(z)\right), \quad \mathcal{L}_2(z) = \begin{pmatrix} L_{p+1}(z) \\ \vdots \\ L_{2p}(z) \end{pmatrix}$$

and $d_L(z) = \operatorname{diag}[z^{-\mu_1}, \dots, z^{-\mu_p}]$, $d_R(z) = \operatorname{diag}[z^{\mu_{p+1}}, \dots, z^{\mu_{2p}}]$.

Then the left ($\lambda_1 \geq \dots \geq \lambda_p$) and right ($\rho_1 \leq \dots \leq \rho_p$) partial indices and the factors ($l_\pm(z)$, $r_\pm(z)$) of the respective factorizations of the matrix polynomial $a(z)$ are defined by the formulas

$$\lambda_1 = -\mu_1, \dots, \lambda_p = -\mu_p, \quad \rho_1 = \mu_{p+1}, \dots, \rho_p = \mu_{2p}, \tag{5}$$

$$l_-(z) = z^{\varkappa+1}\Delta_-(z)d_L^{-1}(z)\mathcal{R}_1^{-1}(z), \quad l_+(t) = z^{-\varkappa-1}\Delta_-^{-1}(z)a(z)\mathcal{R}_1(z), \tag{6}$$

$$r_-(z) = \Delta_-(z)\mathcal{L}_2^{-1}(z), \quad r_+(z) = \Delta_-^{-1}(z)d_R^{-1}(z)\mathcal{L}_2(z)a(z). \tag{7}$$

In the statement of this theorem, we have corrected the misprints appeared in the formulas for the factors $l_+(z)$, $r_+(z)$ in [11, Theorem 3.2].

Let us list the basic steps of the presented factorization algorithm.

1. Calculation of the Laurent coefficients c_j, $-\varkappa \leq j \leq \varkappa$, for the rational matrix functions $\Delta_-^{-1}(z)a(z)$.

   Here $\varkappa = \operatorname{ind}_{\mathbb{T}} \det a(z)$ is a number of zeros of $\det a(z)$ in open disc $|z| < 1$.

   Finding $\varkappa$ and constructing the factorization of scalar polynomial $\Delta(z)$ can be considered as explicit procedures. Now calculation of the Laurent coefficient c_j using recurrence relations requires a finite number of operations.

2. Calculation of the indices for the sequence $c_{-\varkappa}^{\varkappa}$.

   To calculate the indices $\mu_1, \dots, \mu_{2p}$ it is needed to find ranks of the matrices T_k, $-\varkappa \leq j \leq \varkappa$. We can do it by means of linear algebra in a finite number of steps.

3. Calculation of the essential polynomials for the sequence $c_{-\varkappa}^{\varkappa}$.

   For this it is necessary to find bases of the $\ker_{R,L} T_k$, $-\varkappa \leq j \leq \varkappa$. We can do it by means of linear algebra in a finite number of steps.

4. Constructing the factorizations in accordance with Th.2.

   Now this step can be done in an explicit form.

Thus, in accordance with our understanding of the explicit solution of the factorization problem given above, the presented algorithm indeed belongs to this class.

2. Exact Solution of the Factorization Problems for Matrix Polynomials

In this section, we find a condition when the proposed explicit algorithm can be implemented numerically. Due to the instability of the factorization problem, there are two obstacles for doing this.

1. The factorization of scalar polynomial $\Delta(z)$, in general case, can only be constructed approximately.

2. Finding the indices and essential polynomials of the sequence $c_{-\varkappa}^{\varkappa}$ requires calculating ranks and constructing bases of kernels for matrices T_k. Unfortunately, those operations can be unstable.

Thus, in general, the proposed explicit factorization algorithm can not be implemented numerically.

Remark 1. A numerical implementation of the algorithm proposed in [10] meets into the same difficulties.

However, there is still a possibility to implement the algorithm *exactly* by utilising calculations in rational arithmetic. Obviously, we must demand that the coefficients a_j of the original matrix polynomial $a(z)$ must belong the Gaussian field $\mathbb{Q}(i)$ and the factorization of $\Delta(z)$ should be performed *exactly*. In this case the calculations of the Laurent coefficients c_j and finding the indices $\mu_1, \dots, \mu_{2p}$ can also be made exactly.

Now we have to make sure that finding the factorization essential polynomials can also be performed *exactly*. This was not done in [16] and it is the main goal of this work.

In the following theorem we describe the algorithm of finding these essential polynomials and prove that this algorithm can be implemented in the exact arithmetic if entries of the matrices c_j, $-\varkappa \leq j \leq \varkappa$, belong to the field $\mathbb{Q}(i)$.

Theorem 3. *Let $c_{-\varkappa}^{\varkappa} := \{c_{-\varkappa}, \dots, c_0, \dots, c_\varkappa\}$ be a regular sequence of complex $p \times p$ matrices with entries from the field $\mathbb{Q}(i)$. Suppose that the indices $\mu_1, \dots, \mu_{2p}$ of the* *sequence satisfy the condition* $\sum_{j=1}^{p}\mu_j = -\varkappa$, $\sum_{j=p+1}^{2p}\mu_j = \varkappa$, *and the sequence has the factorization essential polynomials. Then these polynomials can be found by calculations in the exact arithmetic.*

Proof. Let us restrict ourselves to considering only the right essential polynomials $R_1(z),\dots,R_{2p}(z)$. Algorithm of finding the factorization essential polynomials is based on the criterion of essentiality (Th. 1).

By Definition 3, these polynomials have zero constant terms: $R_1(0)=\dots=R_p(0)=0$. Hence, to construct the factorization essential polynomials $R_1(z),\dots,R_p(z)$ we must select first p vector polynomials $R_j(z)\in\mathcal{N}^R_{\mu_j+1}$, $j=1,\dots,p$, such that the $2p\times 2p$ matrix

$$\Lambda_R =$$
$$= \left(\begin{array}{ccc|ccc} \widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\} & \cdots & \widetilde{\sigma}_R\{z^{-\varkappa-1}R_p(z)\} & \widetilde{\sigma}_R\{z^{-\varkappa-1}R_{p+1}(z)\} & \cdots & \widetilde{\sigma}_R\{z^{-\varkappa-1}R_{2p}(z)\} \\ \hline 0 & \cdots & 0 & R_{p+1}(0) & \cdots & R_{2p}(0) \end{array}\right).$$

will be invertible or, in other words, the $p\times p$ submatrices

$$\Lambda_{11} = \left(\widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\}\cdots\widetilde{\sigma}_R\{z^{-\varkappa-1}R_p(z)\}\right), \quad \Lambda_{22} = (R_{p+1}(0)\cdots R_{2p}(0))$$

will be invertible.

Now it will be convenient to introduce the distinct indices and to assign them the respective multiplicities. Moreover, it is necessary to highlight the border index μ_p. Let $\nu_1<\dots<\nu_s$ be the distinct indices of the sequence $c^{\varkappa}_{-\varkappa}$ and $\kappa_1,\dots,\kappa_s$ their multiplicities. Let the index μ_p coincides with ν_t.

We will use induction by a number of indices $\nu_1,\dots,\nu_t$. First we select the factorization essential polynomials $R_1(z),\dots,R_{\kappa_1}(z)$ corresponding to the first index ν_1 of κ_1-multiplicity. They are the generating polynomials of vectors forming a basis of $\ker T_{\nu_1+1}$. Since there are the factorization essential polynomials, there exist a basis of $\mathcal{N}_{\nu_1+1}\cong\ker T_{\nu_1+1}$ such that $R_1(z)=z\widetilde{R}_1(z),\dots,R_{\kappa_1}(z)=z\widetilde{R}_{\kappa_1}(z)$ for some polynonials $\widetilde{R}_1(z),\dots,\widetilde{R}_{\kappa_1}(z)$ from the space $\widetilde{\mathcal{N}}_{\nu_1+1}\cong\ker\widetilde{T}_{\nu_1+1}$, where the matrix $\widetilde{T}_{\nu_1+1}$ is obtained from T_{ν_1+1} by deleting of the first p columns, i.e. by deleting the first block column of the matrix. It is easily seen that corresponding vectors $\widetilde{R}_1,\dots,\widetilde{R}_{\kappa_1}$ form a basis $\ker\widetilde{T}_{\nu_1+1}$. Thus, by virtue the essentiality criterion (Th.1) there exists a basis of $\ker\widetilde{T}_{\nu_1+1}$ such that the $2p\times\kappa_1$ submatrix

$$\begin{pmatrix} \tilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\} & \cdots & \tilde{\sigma}_R\{z^{-\varkappa-1}R_{\kappa_1}(z)\} \\ 0 & \cdots & 0 \end{pmatrix}$$

of Λ_R has the rank is equal to $\varkappa_1$. In fact, it is easy to show that this condition is fulfilled for any choice of a basis $\widetilde{R}_1,\dots,\widetilde{R}_{\kappa_1}$. Since the entries of the matrices c_j belong to $\mathbb{Q}(i)$, the construction of this basis and calculation of the rank can be done in the exact arithmetic.

Thus, we can exactly construct the first κ_1 polynomials $R_1(z),\dots,R_{\kappa_1}(z)$ such that $R_1(0)=0,\dots,R_{\kappa_1}(0)=0$, entries of the $p\times\kappa_1$ matrix

$$\left(\widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\}\cdots\widetilde{\sigma}_R\{z^{-\varkappa-1}R_{\kappa_1}(z)\}\right)$$

belong to $\mathbb{Q}(i)$, and this matrix has the rank equal to κ_1.

Now we repeat these considerations for the other indices $\nu_2,\dots,\nu_t$. Assume first that $\mu_p<\mu_{p+1}$. Recall that $\nu_t=\mu_p$ and has the multiplicity κ_t. In this case $\kappa_1+\dots+\kappa_t$ coincides with the number of the indices $\mu_1,\dots,\mu_p$, that is $\kappa_1+\dots+\kappa_t=p$.

Suppose that we construct the polynomials

$$R_1(z),\dots,R_{\kappa_1}(z);R_{\kappa_1+1}(z),\dots,R_{\kappa_1+\kappa_2}(z);\dots;R_{\kappa_1+\dots+\kappa_{j-1}+1}(z),\dots,R_{\kappa_1+\dots+\kappa_j}(z)$$

corresponding to the indices $\nu_1, \dots, \nu_j$, $2 \le j \le \nu_{t-1}$, such that $R_1(0) = 0, \dots, R_{\kappa_1+\dots+\kappa_j}(0) = 0$, entries of the $p \times (\kappa_1 + \dots + \kappa_j)$ matrix

$$\left(\widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\} \cdots \widetilde{\sigma}_R\{z^{-\varkappa-1}R_{\kappa_1+\dots+\kappa_j}(z)\}\right)$$

belong to $\mathbb{Q}(i)$, and this matrix has the rank equal to $\kappa_1 + \dots + \kappa_j$.

Let us define the polynomials $R_{\kappa_1+\dots+\kappa_j+1}(z), \dots, R_{\kappa_1+\dots+\kappa_{j+1}}(z)$ corresponding to the index ν_{j+1} of the multiplicity κ_{j+1}. These polynomials belong to the space $\mathcal{N}_{\nu_{j+1}+1} \cong \ker T_{\nu_{j+1}+1}$. Let $\widetilde{n}_{j+1}$ is the dimension of the space $\widetilde{\mathcal{N}}_{\nu_{j+1}+1} \cong \ker \widetilde{T}_{\nu_{j+1}+1}$ and the polynomials $Q_1(z), \dots, Q_{\widetilde{n}_{j+1}}(z)$ be a basis of this space. Here $\widetilde{T}_{\nu_{j+1}+1}$ is obtained from $T_{\nu_{j+1}+1}$ by deleting the first block column of the matrix. Hence, $zQ_1(z), \dots, zQ_{\widetilde{n}_{j+1}}(z)$ is a basis of the space $\mathcal{N}_{\nu_{j+1}+1}$.

From this basis , we consequently select the required polynomials. Such a selection we will do consequently. First we choose among $zQ_1(z), \dots, zQ_{\widetilde{n}_{j+1}}(z)$ a polynomial $R_{\kappa_1+\dots+\kappa_j+1}(z)$ such that the matrix

$$\left(\widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\} \cdots \widetilde{\sigma}_R\{z^{-\varkappa-1}R_{\kappa_1+\dots+\kappa_j}(z)\ \widetilde{\sigma}_R\{z^{-\varkappa-1}R_{\kappa_1+\dots+\kappa_j+1}(z)\}\right)$$

has the rank equal to $\kappa_1 + \dots + \kappa_j + 1$. This selection is always possible since the sequence $c_{-\varkappa}^{\varkappa}$ has the factorization essential polynomials. In a similar way, we select the other polynomials $R_{\kappa_1+\dots+\kappa_j+2}(z), \dots, R_{\kappa_1+\dots+\kappa_{j+1}}(z)$ for which the matrix

$$\left(\widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\} \cdots \widetilde{\sigma}_R\{z^{-\varkappa-1}R_{\kappa_1+\dots+\kappa_{j+1}}(z)\}\right)$$

has the rank equal to $\kappa_1 + \dots + \kappa_{j+1}$.

Hence, in the case of $\varkappa_1 + \dots + \varkappa_t = p$, we obtain, by induction, the polynomials $R_1(z), \dots, R_p(z)$, for which the matrix

$$\Lambda_{11} = \left(\widetilde{\sigma}_R\{z^{-\varkappa-1}R_1(z)\} \cdots \widetilde{\sigma}_R\{z^{-\varkappa-1}R_p(z)\}\right)$$

over $\mathbb{Q}(i)$ has the rank equal to p. Thus, in this case, the first p factorization essential polynomials $R_1(z), \dots, R_p(z)$ are exactly constructed.

Now, we build the polynomials $R_{p+1}(z), \dots, R_{2p}(z)$ such that the matrix Λ_{22} is invertible. These polynomials must be sequentially chosen from the spaces $\mathcal{N}_{j+1} \cong \ker T_{j+1}$, $j = \nu_{t+1}, \dots, \nu_s$. It is clear that we can always choose polynomials $R_{p+1}(z), \dots, R_{p+\kappa_{t+1}}(z)$ from the basis $\mathcal{N}_{\nu_{t+1}+1}$, such that vectors $R_{p+1}(0), \dots, R_{p+\kappa_{t+1}}(0)$ are linear independent. Otherwise, the sequence $c_{-\varkappa}^{\varkappa}$ would not have factorization essential polynomials. Repeating these arguments for the indices $\nu_{t+2} \dots, \nu_s$ we arrive to polynomials $R_{p+1}(z), \dots, R_{2p}(z)$ for which the matrix Λ_{22} is invertible. Therefore, in the case of $\mu_p < \mu_{p+1}$, the right factorization essential polynomials can always be found by the *exact* computation.

Let us consider now the case when the border index μ_p satisfies the equality $\mu_p = \mu_{p+1}$, or more precisely, when

$$\mu_1 \le \dots \le \mu_{p-l} = \nu_{t-1} < \mu_{p-l+1} = \dots = \mu_p = \dots = \mu_{p+m} = \nu_t < \mu_{p+m+1} \le \dots \le \mu_{2p}$$

for some $l > 0$, $m > 0$. Then $\kappa_1 + \dots + \kappa_{t-1} = p - l$, $\kappa_t = l + m$, $\kappa_{t+1} + \dots + \kappa_s = p - m$ and $\kappa_1 + \dots + \kappa_t = p + m > p$. The right factorization polynomials $R_1(z), \dots, R_{\kappa_1+\dots\kappa_{t-1}}(z)$ corresponding to the indices $\nu_1, \dots, \nu_{t-1}$ we can construct as above. Recall that κ_t is the number of the right essential polynomials $R_{\kappa_1+\dots\kappa_{t-1}+1}(z), \dots, R_{\kappa_1+\dots\kappa_t}(z)$ corresponding to the index ν_t. They belong to the space $\mathcal{N}_{\nu_t+1} \cong \ker T_{\nu_t+1}$. These polynomials are divided into two type. For the first l polynomials $R_{\kappa_1+\dots\kappa_{t-1}+1}(z), \dots, R_p(z)$, the conditions

$R_{\kappa_1+\cdots\kappa_{t-1}+1}(0)=0,\ldots,R_p(0)=0$ and the invertibility of the matrix Λ_{11} must be fulfilled. The remaining m polynomials $R_{p+1}(z),\ldots,R_{\varkappa_1+\cdots+\varkappa_t}(z)\in\mathcal{N}_{\nu_t+1}$ must be chosen so that the vectors $R_{p+1}(0),\ldots,R_{\varkappa_1+\cdots+\varkappa_t}(0)$ are linearly independent.

The first type polynomials we can construct as above by choosing successively l polynomials from a basis $Q_1(z),\ldots,Q_{n_t+1}$ of the space $\widetilde{\mathcal{N}}_{\nu_t+1}\cong\ker\widetilde{T}_{\nu_t+1}$. The existence of the factorization essential polynomials guarantees that this process can be carried out.

The remaining m polynomials $R_{p+1}(z),\ldots,R_{\varkappa_1+\cdots+\varkappa_t}(z)$ must be chosen from the elements of a basis of the space $\mathcal{N}_{\mu_p+1}\cong\ker T_{\nu_t+1}$ in a way that the rank of the matrix $(R_{p+1}(0)\cdots R_{\varkappa_1+\cdots+\varkappa_t}(0))$ is equal to $m=\varkappa_1+\cdots+\varkappa_t-p$. It is again possible since the sequence $c_{-\varkappa}^{\varkappa}$ possesses the factorization essential polynomials.

By repeating this choice for the spaces $\mathcal{N}_{\nu_j+1}\cong\ker T_{\nu_j+1}$, $j=\nu_{t+1},\ldots,\nu_s$, we obtain the polynomials $R_{p+1}(z),\ldots,R_{2p}(z)$ for which the matrix Λ_{22} is invertible. Then, for the polynomials $R_1(z),\ldots,R_p(z)$, $R_{p+1}(z),\ldots,R_{2p}(z)$, the matrix Λ_R is invertible and these polynomials are the right factorization essential polynomials. To evaluate these polynomials, we have solved block Toeplitz systems with the coefficients belonging to $\mathbb{Q}(i)$ and have found the ranks of matrices with entries from this field. All such operations can be performed exactly.

To obtain the left factorization essential polynomials, we can carry out similar construction with the sequence of left kernels of matrices T_k, $-\varkappa\le k\le\varkappa$, or can apply a conformance procedure (see [17], Def. 5.3). This procedure can be also fulfilled *exactly*. The conformance procedure that is used to construct the left factorization essential polynomials, is described in [17].

□

After finding the indices and factorization essential polynomials, we can exactly construct the Wiener–Hopf factorizations using the formulas (5) – (7).

3. Pseudo-Code for an Exact Constructing the Right Factorization Essential Polynomials

The full variant of the pseudo-code for the algorithm of simultaneous construction of the left and right factorizations is given in [18]. However, if only one type of the factorization is needed (for instance, the left factorization), using the full algorithm leads to a significant increasing in execution time. For this reason, in this section we give the pseudo-code for construction of the left factorization only. For simplicity, here we restrict ourselves to the case when $\mu_p<\mu_{p+1}$.

Algorithm. Indices and right factorization essential polynomials of a sequence

---

Input. The sequence $c_{-\varkappa}^{\varkappa}:=\{c_{-\varkappa},\ldots,c_0\ldots,c_\varkappa\}$, $c_j\in\mathbb{Q}^{p\times p}(i)$.
Output. The indices $\mu_1,\ldots,\mu_{2p}$ and the matrix of the right factorization essential polynomials, $\mathcal{R}_1:=\big(R_1(z)\cdots R_p(z)\big)$,

---

1. find the distinct indices $\nu_1,\ldots,\nu_s$, their multiplicities $\kappa_1,\ldots,\kappa_s$, form the indices $\mu_1,\ldots,\mu_{2p}$, and the number t such that $\mu_p=\nu_t$
2. find the polynomials $\widetilde{R}_1(z),\ldots,\widetilde{R}_{\kappa_1}(z)$ forming a basis of the space $\widetilde{\mathcal{N}}_{\nu_1+1}\cong\ker\widetilde{T}_{\nu_1+1}$, define the matrix $\mathcal{R}_1:=\big(z\widetilde{R}_1(z)\cdots z\widetilde{R}_{\kappa_1}(z)\big)$
3. form the matrix $\sigma_{11}:=\big(\sigma_R\{z^{-\varkappa}\widetilde{R}_1(z)\}\cdots\sigma_R\{z^{-\varkappa}\widetilde{R}_{\kappa_1}(z)\}\big)$
4. **for** $j=2,\ldots,t$ **do**

5. find a basis $\widetilde{Q}_1(z), \ldots, \widetilde{Q}_{\widetilde{n}_j}(z)$ of the space $\widetilde{\mathcal{N}}_{\nu_j+1} \cong \ker \widetilde{T}_{\nu_j+1}$

6. **for** $k = 1, \ldots, \widetilde{n}_j$ **do**

7. form the matrix $\sigma_2 := \begin{pmatrix}\sigma_{11} & \sigma_R\{z^{-\varkappa}\widetilde{Q}_k(z)\}\end{pmatrix}$

8. **if** $\operatorname{rank}\sigma_2 = \operatorname{rank}\sigma_{11} + 1$ **then**

9. $\sigma_{11} := \sigma_2$

10. $\mathcal{R}_1 := \Big(\mathcal{R}_1 \;\; z\widetilde{Q}_k(z)\Big)$

11. **end if**

12. **end do**

13. **end do**

14. **if** $\operatorname{rank}\sigma_{11} \neq p$ **then**

15. **print** "The factorization essential polynomials were not constructed. The factorization process is interrupted"

16. **stop**

17. **end if**

18. form $D(z) = \begin{pmatrix} z^{\mu_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & z^{\mu_{2p}} \end{pmatrix}$.

19. **return** $\mu_1, \ldots, \mu_{2p}$, $\mathcal{R}_1(z)$

Now by formulas (5), (6) we can construct the left factorization of a matrix polynomial.

4. Numerical Example

Based on the proposed algorithm, a procedure ExactFEP was developed, which is the main part of the ExactMPF package in Maple. The package is designed for the exact solution of the factorization problem for matrix polynomials. To access ExactMPF use the commands

```
> read("ExactMPF.txt");
> with(ExactMPF);
> with(LinearAlgebra);
```

To obtain the factorizations of $a(z)$ we run the module SolverExactMPF with the argument $a(z)$:

```
> lplus, dl, lminus, rminus, dr, rplus := SolverExactMPF(a):
```

The module SolverExactMPF returns the factors lplus, dl, lminus of the left factorization and the factors rminus, dr, rplus of the right factorization.

Let us give an example of using this package.

Example 1. Consider

$$a(z) := \begin{pmatrix} 36z^2+17z-14 & z^4-z^2+3z-1 & z+10 \\ 0 & z^2+13z+15 & z^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The module SolverExactMPF gives in this case the following expression for the factors of $a(z)$:

```
> lplus; dl; lminus;
```

$$\begin{bmatrix} 36 & z+10 & z^4-z^2+3z-1 \\ 0 & z^2 & z^2+13z+15 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1+\frac{17}{36z}-\frac{7}{18z^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The executing time is 0,500 seconds when computations were performed on a home desktop computer HP with Intel(R) Core(TM)i3-415T CPU, 3.00 GHz, 4G RAM, operating system Windows 10.

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ЗАМЕЧАНИЕ ОБ АЛГОРИТМЕ ТОЧНОЙ ФАКТОРИЗАЦИИ ДЛЯ МАТРИЧНЫХ МНОГОЧЛЕНОВ

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Существуют два основных препятствия для широкого использования метода факторизации Винера – Хопфа для матриц-функций, используемых для решения векторных краевых задач Римана. Первое препятствие связано с отсутствием общего явного метода факторизации в матричном случае, хотя для конкретных классов матричных функций могут существовать явные (конструктивные) методы факторизации. Второе препятствие является следствием того, что факторизация матриц-функций, вообще говоря, является неустойчивой по отношению к малому возмущению исходной функции. В результате последнего, реализация любого конструктивного алгоритма, даже если он существует для данной матрицы-функции, на практике не может быть осуществлена. Более того, разрабатывая явные методы, авторы часто не анализируют его численную реализацию, неявно предполагая, что все шаги предложенного конструктивного алгоритма могут быть выполнены точно. В предлагаемой работе мы продолжаем изучение связи между явным и точным решениями задачи факторизации в классе матричных многочленов. Основная цель – получить алгоритм точного вычисления так называемых индексов и существенных многочленов конечной последовательности матриц. Это краеугольный камень проблемы точной факторизации матричных многочленов.

Ключевые слова: факторизация Винера – Хопфа; теплицевы матрицы; существенные многочлены последовательности.

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