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INVESTIGATION OF THE UNIQUENESS SOLUTION OF THE SHOWALTER–SIDOROV PROBLEM FOR THE MATHEMATICAL HOFF MODEL. PHASE SPACE MORPHOLOGY

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The study of the phase space morphology of the mathematical model deformation of an I-beam, which lies on smooth Banach manifolds with singularities (k-Whitney assembly) depending on the parameters of the problem, is devoted to the paper. The mathematical model is studied in the case when the operator at time derivative is degenerate. The study of the question of non-uniqueness of the solution of the Showalter–Sidorov problem for the Hoff model in the two-dimensional domain is carried out on the basis of the phase space method, which was developed by G.A. Sviridyuk. The conditions of non-uniqueness of the solution in the case when the dimension of the operator kernel at time derivative is equal to 1 or 2 are found. Two approaches for revealing the number of solutions of the Showalter–Sidorov problem in the case when the dimension of the operator kernel at time derivative is equal to 2 are presented. Examples illustrating the non-uniqueness of the solution of the problem on a rectangle are given.

Keywords: Sobolev type equations; Showalter–Sidorov problem; phase space method; Whitney assemblies; the Hoff equation; non-uniqueness of solutions.

Introduction

Extensive class of models of mathematical physics is based on semilinear non-classical equations or systems of partial derivative equations unsolved with respect to the time derivative

$$L\dot{u} = Mu + N(u), \tag{1}$$

which are commonly called Sobolev type equations. Equations of this class and initial problems for them cannot be investigated by classical methods due to the possible degeneracy of the operator at the higher derivative, so their investigation requires the development of new and modifications of already known methods of investigation [1–5]. Let us consider a mathematical model based on the Sobolev type equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^∞. Let us consider the Hoff model [6]

$$(\mu + \Delta)u_t = \alpha u + \beta u^3, \; x \in \Omega, t \in (0, T), \tag{2}$$

with the Dirichlet condition

$$u(x,t) = 0, \; (x,t) \in \partial\Omega \times (0,T). \tag{3}$$

The Hoff model describes the deformation dynamics of an I-beam. The unknown function $u = u(x,t), \; x \in \Omega, \; t \in (0,T)$, has the physical meaning of the deflection of the beam from the equilibrium position. The parameter $\mu \in \mathbb{R}$ characterises the longitudinal load on the beam and the parameters $\alpha, \beta \in \mathbb{R}$ characterise the material properties of the beam. Studies for equation (2) on graphs are presented in [7, 8], on manifolds in [9].

One of the first to study the initial boundary value problem for equation (2) was N.A. Sidorov [10]. In this work, the principal insolvability of the Cauchy problem ($u(x,0) - u_0(x) = 0$) at an arbitrary initial value in the case of degeneracy of equation (2) was noted. Consideration of the Showalter–Sidorov condition [11]

$$L\left(u(x,0) - u_0(x)\right) = 0 \tag{4}$$

allows one to avoid difficulties in solving the Cauchy problem, but non-uniqueness of the solution of problems (2) – (4) is possible [12].

The questions of non-uniqueness of solutions of equations and systems of equations reduced to semilinear equations of the form (1) with the Showalter–Sidorov condition (4) and the connection of non-uniqueness of the solution with the existence of Whitney assemblies and folds in the phase space of equation (1) were devoted to the following works: T.A. Bokareva and G.A. Sviridyuk for the model of nerve impulse propagation in the membrane and for the model of autocatalytic reaction with diffusion showed the existence of 2-Whitney assemblies and 1-Whitney folds, respectively [13], A.F. Gilmutdinova for the Plotnikov mathematical model revealed the conditions for the existence of non-uniqueness of the solution [14].

To take advantage of earlier results obtained in [12–14] for semilinear abstract Sobolev type equations, let us reduce the problem (2), (3) to equation (1). For this purpose, we assume $\mathfrak{U} = \overset{\circ}{W}{}_2^1(\Omega)$, $\mathfrak{F} = W_2^{-1}(\Omega)$. The operators L, M are defined by the formulas

$$\begin{aligned} \langle Lu, v\rangle &= \langle \mu u + \Delta u, v\rangle \, \forall u, v \in \mathfrak{U}, \\ \langle Mu, v\rangle &= \langle \alpha u, v\rangle \, \forall u, v \in \mathfrak{U}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(\Omega)$. We note that the operator $L \in \mathfrak{L}(\mathfrak{U}, \mathfrak{F})$, $M \in \mathfrak{L}(\mathfrak{U}, \mathfrak{F})$. The spectrum $\sigma(L)$ of the operator L is real, discrete, finite-edge, and condensed to $-\infty$. Now let us construct the operator N

$$\langle N(u), v\rangle = \left\langle \beta u^3, v\right\rangle \;\; \forall u, v \in \mathfrak{U}.$$

By virtue of Gelder's inequality

$$|\langle N(u), v\rangle| \le |\beta \| u\|^3_{L_4(\Omega)} \|v\|_{L_4(\Omega)},$$

operator $N : L_4(\Omega) \to (L_4(\Omega))^* \cong L_{4/3}(\Omega)$. When $n \le 4$ the embedding $\overset{\circ}{W}{}_2^1(\Omega) \hookrightarrow L_4(\Omega)$ is dense and continuous, and hence the embedding $L_{4/3}(\Omega) \hookrightarrow W_2^{-1}(\Omega)$. Thus, the action of the operator $N : \mathfrak{U} \to \mathfrak{F}$. By virtue of the given operators, the Showalter–Sidorov problem (4) for equation (2) will take the form

$$(\mu + \Delta)\left(u(x,0) - u_0(x)\right) = 0, \;\; x \in \Omega. \tag{5}$$

G.A. Sviridyuk and his successors developed a method [1, 15], based on the study of the morphology of the set of admissible initial values $\mathfrak{B}$

$$\mathfrak{B} = \{u \in \mathfrak{U} : \langle (\mathbb{I} - Q)\left(Mu + N(u)\right)\rangle = 0\}, \tag{6}$$

understood as the phase space of equation (1). In [16] the projector

$$Q = \mathbb{I} - \sum_{k=1}^{m} \langle \cdot, \varphi_l \rangle \, \varphi_l$$

was constructed and it is shown that the phase space (6) for model (2), (3) take the form

$$\mathfrak{B} = \left\{ u \in \mathfrak{U} : \int_{\Omega} (\alpha + \beta u^2) u \varphi_l dx = 0 \right\}, \qquad (7)$$

where φ_l are the eigenfunctions of the homogeneous Dirichlet problem of the operator L.

In the case when the phase space of the model (2), (3) has singularities, the non-uniqueness of the solution of the Showalter–Sidorov problem arises, and the simplicity of the phase space of the model (2), (3) implies the singularity of the solution. In [16] it was shown that the phase space of equation (2) is a simple Banach C^∞-manifold in case $\alpha\beta > 0$, in case $\alpha\beta < 0$ the phase space of equation (2) may contain a 2-Whitney assembly [17]. In [18], conditions on the parameters α, β, were found under which the phase space of equation (2) has singularities in case $\dim \ker(\mu + \Delta) = 1$.

As the paper, in addition to theoretical studies, also contains the results of numerical experiments, it is necessary to mention the Galerkin method, which is the basis for the computational experiments. Obtaining an analytical solution of Sobolev type equations (1) is not always possible, so the construction of algorithms for numerical methods is in demand. For degenerate semilinear equations, the Galerkin method is the most appropriate one, since it allows to incorporate the degeneracy of the equation for some parameters. Using the Galerkin method, approximate solutions of the problem are constructed, whose coefficients satisfy the system of algebro-differential equations with appropriate initial conditions [19–21].

The purpose of this study is to investigate the model (2), (3) and to identify the conditions imposed on the parameters α, β, under which the phase space has singularities and there are several solutions to the Showalter–Sidorov problem (2), (3), (5) at $\dim \ker(\mu + \Delta) = 1$ and $\dim \ker(\mu + \Delta) = 2$.

1. Features of Phase Space

Let us study the morphology of the phase space of the model (2), (3) in the case $\Omega \subset \mathbb{R}^2$. Let us find the conditions imposed on the parameters of the equation α, β, in which the phase space has singularities and it follows that the solution to the Showalter–Sidorov problem is non-unique (2), (3), (5).

Let us consider the homogeneous Dirichlet problem for the Laplace operator $(-\Delta)$. Let us denote by $\{\lambda_{k_1,\, k_2}\}$ the family of eigenvalues of the problem under consideration. In case

a) if $\mu = \lambda_{k_1,\, k_2}$ and $k_1 = k_2$, then $\dim \ker(\mu + \Delta) = 1$ and the considered eigenvalue $\lambda_{k_1,\, k_2}$ corresponds to one eigenfunction $\varphi_{k_1,\, k_2}(x, y)$, then u can be represented as $u = s_1 \varphi_{k_1,\, k_2} + u^\perp$, $u^\perp \in \mathfrak{U}^\perp = \{u \in L_4(\Omega) : \langle u, \varphi_{k_1,\, k_2} \rangle = 0\}$;

b) if $\mu = \lambda_{k_1,\, k_2}$ and $k_1 \neq k_2$, then $\dim \ker(\mu + \Delta) = 2$ and the considered eigenvalue $\lambda_{k_1,\, k_2}$ corresponds to two eigenfunction $\varphi_{k_1,\, k_2}(x, y)$ and $\varphi_{k_2,\, k_1}(x, y)$, then u can be represented as $u = s_1\varphi_{k_1,\ k_2} + s_2\varphi_{k_2,\ k_1} + u^\perp$, $u^\perp \in \mathfrak{U}^\perp = \{u \in L_4(\Omega) : \langle u, \varphi_{k_1,\ k_2}\rangle = 0, \langle u, \varphi_{k_2,\ k_1}\rangle = 0\}$.

Example 1. As an example, consider $\Omega = (0, l_1) \times (0, l_2)$, then the Dirichlet condition (3) takes the following form:

$$u(0, y, t) = u(l_1, y, t) = 0, u(x, 0, t) = u(x, l_2, t) = 0.$$

At the same time

$$\varphi_{k_1,\ k_2}(x, y) = \sqrt{\frac{4}{l_1 l_2}} \sin\frac{\pi k_1 x}{l_1} \sin\frac{\pi k_2 y}{l_2}, \lambda_{k_1,\ k_2} = \left(\frac{\pi k_1}{l_1}\right)^2 + \left(\frac{\pi k_2}{l_2}\right)^2,\ k = 1, 2, \ldots$$

In case $l_1 = l_2 = \pi$,

a) if $\mu = \lambda_{1,1} = 2$ ($k_1 = k_2 = 1$), then the considered eigenvalue $\lambda_{1,1}$ corresponds to one eigenfunction $\varphi_{1,1}(x, y) = \dfrac{2}{\pi}\sin x \sin y$ and $\dim\ker(\mu + \Delta) = 1$;

b) if $\mu = \lambda_{1,2} = 5$ ($k_1 \neq k_2$), then the considered eigenvalue $\lambda_{1,2}$ corresponds to two eigenfunctions $\varphi_{1,\ 2}(x, y) = \dfrac{2}{\pi}\sin x \sin 2y$, $\varphi_{2,\ 1}(x, y) = \dfrac{2}{\pi}\sin 2x \sin y$ and $\dim\ker(\mu + \Delta) = 2$.

In case a) $u = s_1\varphi_{k_1,\ k_2} + u^\perp$, then set $\mathfrak{B}$ C^∞-diffeomorphic to the set

$$\mathfrak{B}_1 = \Big\{(s_1, u^\perp) \in \mathbb{R} \times L_4(\Omega) : s_1^3 \|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)} + 3s_1^2 \iint_\Omega \varphi^3_{k_1,\ k_2} u^\perp\ dxdy\ + \\ + s_1 \left(3\iint_\Omega \varphi^2_{k_1,\ k_2}(u^\perp)^2\ dxdy + \alpha\beta^{-1}\right) + \iint_\Omega \varphi_{k_1,\ k_2}(u^\perp)^3\ dxdy = 0\Big\}. \tag{8}$$

In case b) $u = s_1\varphi_{k_1,\ k_2} + s_2\varphi_{k_2,\ k_1} + u^\perp$, then set $\mathfrak{B}$ C^∞-diffeomorphic to the set

$$\mathfrak{B}_2 = \big\{(s_1, s_2, u^\perp) \in \mathbb{R}^2 \times L_4(\Omega) : \\ \beta\iint_\Omega \varphi_{k_2,\ k_1}(u^\perp)^3\ dxdy + \beta\iint_\Omega \varphi_{k_1,\ k_2}(u^\perp)^3\ dxdy + \alpha s_1 + \alpha s_2 + \\ + 3\beta s_1 \iint_\Omega \varphi^2_{k_1,\ k_2}(u^\perp)^2\ dxdy + 3\beta s_2 \iint_\Omega \varphi_{k_1,\ k_2}\varphi_{k_2,\ k_1}(u^\perp)^2\ dxdy + \\ + 3\beta s_1 \iint_\Omega \varphi_{k_1,\ k_2}\varphi_{k_2,\ k_1}(u^\perp)^2\ dxdy + 3\beta s_2 \iint_\Omega \varphi^2_{k_2,\ k_1}(u^\perp)^2\ dxdy + \\ + 3\beta s_1^2 \iint_\Omega \varphi^3_{k_1,\ k_2} u^\perp\ dxdy + 3\beta s_2^2 \iint_\Omega \varphi_{k_1,\ k_2}\varphi^2_{k_2,\ k_1} u^\perp\ dxdy + \\ + 3\beta s_2^2 \iint_\Omega \varphi^3_{k_2,\ k_1} u^\perp\ dxdy + 3\beta s_1^2 \iint_\Omega \varphi^2_{k_1,\ k_2}\varphi_{k_2,\ k_1} u^\perp\ dxdy + \\ + 3\beta s_1^2 s_2 \iint_\Omega \varphi^3_{k_1,\ k_2}\varphi_{k_2,\ k_1}\ dxdy + 3\beta s_1 s_2^2 \iint_\Omega \varphi^2_{k_1,\ k_2}\varphi^2_{k_2,\ k_1}\ dxdy + \\ + 3\beta s_1^2 s_2 \iint_\Omega \varphi^2_{k_1,\ k_2}\varphi^2_{k_2,\ k_1}\ dxdy + 3\beta s_1 s_2^2 \iint_\Omega \varphi_{k_1,\ k_2}\varphi^3_{k_2,\ k_1}\ dxdy + \\ \beta s_1^3 \|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)} + \beta s_2^3 \iint_\Omega \varphi_{k_1,\ k_2}\varphi^3_{k_2,\ k_1}\ dxdy + \\ + \beta s_2^3 \|\varphi_{k_2,\ k_1}\|^4_{L_4(\Omega)} + \beta s_1^3 \iint_\Omega \varphi^3_{k_1,\ k_2}\varphi_{k_2,\ k_1}\ dxdy = 0\Big\}. \tag{9}$$

The sets $\mathfrak{B}_1$ and $\mathfrak{B}_2$ describe the phase space of the model (2), (3) in the case of $k_1 = k_2$ and $k_1 \neq k_2$, respectively. Let us give a definition of the phase space containing the k-Whitney assembly.

Definition 1. [14] *Let $\mathfrak{V}$ be a Banach space, function $G \in C^{\infty}(\mathbb{R} \times \mathfrak{V}; \mathbb{R})$. The equation $G(s, v) = 0$ defines a k-Whitney assemblys over the open set $\mathfrak{V}' \subset \mathfrak{V}$ if there exist functions $g_0, g_1, ..., g_k \in C^{\infty}(\mathfrak{V}'; \mathbb{R})$ such that this equation is equivalent to the equation*

$$0 = g_0(v) + g_1(v)s + ... + g_k(v)s^k + s^{k+1} \quad \forall v \in \mathfrak{V}'.$$

Let us formulate a theorem about the structure of the phase space of the model (2), (3).

Theorem 1.

(i) Let $\alpha\beta > 0$, then the phase space of the model (2), (3) is a simple Banach C^{∞}-manifold modeled by a subspace complementary to $\ker(\mu + \Delta)$.

(ii) Let $\alpha\beta < 0$ and $k_1 = k_2$, then the set $\mathfrak{B}_1$ forms a Whitney 2-assembly.

(iii) Let $\alpha\beta < 0$ and $k_1 \neq k_2$, then the set $\mathfrak{B}_2$ forms a k-Whitney assembly.

Proof. The proof of item 1 is given in the work [16], the proof of item 2 is given in the work [17]. The validity of item 3 follows from the construction of the set $\mathfrak{B}_2$ (9) and Definition 1. The set $\mathfrak{B}_2$ contains the k-Whitney assembly, $k = 2 \ldots 8$. The degree of assembly depends on the model parameters and the type of domain Ω.

□

The equation defining the set $\mathfrak{B}_1$ is a cubic equation of general form

$$as_1^3 + bs_1^2 + cs_1 + d = 0. \tag{10}$$

Let's define

$$\begin{aligned} Q_1(u) &= \left(\frac{3ac - b^2}{9a^2}\right)^3 + \frac{1}{4}\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right)^2, \\ R_1(u) &= s_1^2 \|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)} + 2s_1 \iint\limits_{\Omega} \varphi^3_{k_1,\ k_2} u^{\perp}\, dxdy + \iint\limits_{\Omega} \varphi^2_{k_1,\ k_2} (u^{\perp})^2\, dxdy, \end{aligned} \tag{11}$$

where

$$\begin{aligned} &a = \|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)},\ b = 3 \iint\limits_{\Omega} \varphi^3_{k_1,\ k_2} u^{\perp}\, dxdy, \\ &c = 3 \iint\limits_{\Omega} \varphi^2_{k_1,\ k_2} (u^{\perp})^2\, dxdy + \alpha\beta^{-1}, d = \iint\limits_{\Omega} \varphi_{k_1,\ k_2} (u^{\perp})^3 dxdy, \end{aligned} \tag{12}$$

and consider the following sets

$$\begin{aligned} &(\mathfrak{U}_1)^{\perp}_0 = \{u \in \mathfrak{U}^{\perp} : R_1(u) = 0\}, \\ &(\mathfrak{U}_1)^{\perp}_+ = \{u \in \mathfrak{U}^{\perp} : Q_1(u) > 0\}, (\mathfrak{U}_1)^{\perp}_- = \{u \in \mathfrak{U}^{\perp} : Q_1(u) < 0\}. \end{aligned}$$

Theorem 2. *Let $\alpha\beta < 0$. Then*
(i) for any $u \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_+$ there is one solution to the equation (10);
(ii) for any $u \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_+ \cap (\mathfrak{U}_1)^{\perp}_0$ there are two solutions equations (10);
(iii) for any $u \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_-$ there are three solutions to the equation (10).

Proof. The theorem is valid due to the Cardano formulas (11), (12) for the equation (10).

□

Theorem 3. *Let $\alpha\beta < 0$ and $\mu = \lambda_{k_1,\ k_2}, k_1 = k_2$. Then*
(i) for any $u_0 \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_{-}$ there are three solutions to the problem (2), (3), (5);
(ii) for any $u_0 \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_{+} \cap (\mathfrak{U}_1)^{\perp}_{0}$ there are one or two solutions to the problem (2), (3), (5);
(iii) for any $u_0 \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_{+}$ there is only one solution to the problem (2), (3), (5).

Proof. *(i)* Let's take point $u_0 \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_{-}$. According to Theorem 2, there are three solutions s_1^1, s_1^2, s_1^3 to the equation (10), which means that the point u_0 serves as the image of three points $u_0^1 = s_1^1\varphi_{k_1,\ k_2} + u^{\perp} \in \mathfrak{B}_1$, $u_0^2 = s_1^2\varphi_{k_1,\ k_2} + u^{\perp} \in \mathfrak{B}_1$, $u_0^3 = s_1^3\varphi_{k_1,\ k_2} + u^{\perp} \in \mathfrak{B}_1$. According to the Theorem on the existence of a solution to the Showalter–Sidorov problem [22] the problem (2), (3), (5) should have three different solutions.

(ii) Let's take point $u_0 \in (\mathfrak{U}_1)^{\perp} \cap (\mathfrak{U}_1)^{\perp}_{+} \cap (\mathfrak{U}_1)^{\perp}_{0}$. According to Theorem 2, there are two s_1^1, s_1^2 or one s_1 solutions to the equation (10), which means that the point u_0 serves as the image of three points $u_0 = s_1\varphi_{k_1,\ k_2} + u^{\perp} \in \mathfrak{B}_1$ or for two points $u_0^1 = s_1^1\varphi_{k_1,\ k_2} + u^{\perp} \in \mathfrak{B}_1$, $u_0^2 = s_1^2\varphi_{k_1,\ k_2} + u^{\perp} \in \mathfrak{B}_1$.

(iii) Let's take point $u_0 \in \mathfrak{U}^{\perp} \cap \mathfrak{U}^{\perp}_{+}$. According to Theorem 2, there is only one solution s_1 to the equation (10), which means that the point u_0 serves as the image of only one point $u_0 = s_1\varphi_{k_1,\ k_2} + u^{\perp} \in \mathfrak{B}_1$. According to the Theorem on the existence of a solution to the Showalter–Sidorov problem the problem (2), (3), (5) should have only one solution.

□

Remark 1. In the case of $\Omega \subset \mathbb{R}$, similar conditions were obtained in the work [18].

Let's move on to studying case b), in which $\mu = \lambda_{k_1,\ k_2}$ $(k_1 \neq k_2)$ и $\dim\ker(\mu+\Delta) = 2$. To do this, we construct the set (9) in an equivalent form:

$$\mathfrak{B}_2 = \Big\{(s_1, s_2, u^{\perp}) \in \mathbb{R}^2 \times L_4(\Omega) :$$

$$\begin{aligned}
&\alpha s_1 + \beta s_1^3 \|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)} + \\
&+ \beta s_2^3 \iint\limits_{\Omega} \varphi_{k_1,\ k_2}\varphi^3_{k_2,\ k_1}\,dxdy + \beta \iint\limits_{\Omega} \varphi_{k_1,\ k_2}(u^{\perp})^3\,dxdy + \\
&+ 3\beta s_1^2 s_2 \iint\limits_{\Omega} \varphi^3_{k_1,\ k_2}\varphi_{k_2,\ k_1}\,dxdy + 3\beta s_1 s_2^2 \iint\limits_{\Omega} \varphi^2_{k_1,\ k_2}\varphi^2_{k_2,\ k_1}\,dxdy + \\
&+ 3\beta s_1^2 \iint\limits_{\Omega} \varphi^3_{k_1,\ k_2}u^{\perp}\,dxdy + 3\beta s_1 \iint\limits_{\Omega} \varphi^2_{k_1,\ k_2}(u^{\perp})^2\,dxdy + \\
&+ 3\beta s_2^2 \iint\limits_{\Omega} \varphi_{k_1,\ k_2}\varphi^2_{k_2,\ k_1}u^{\perp}\,dxdy + 3\beta s_2 \iint\limits_{\Omega} \varphi_{k_1,\ k_2}\varphi_{k_2,\ k_1}(u^{\perp})^2\,dxdy = 0,
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\alpha s_2 + \beta s_2^3 \|\varphi_{k_2,\ k_1}\|^4_{L_4(\Omega)} + \\
&+ \beta s_1^3 \iint\limits_{\Omega} \varphi^3_{k_1,\ k_2}\varphi_{k_2,\ k_1}\,dxdy + \beta \iint\limits_{\Omega} \varphi_{k_2,\ k_1}(u^{\perp})^3\,dxdy + \\
&+ 3\beta s_1^2 s_2 \iint\limits_{\Omega} \varphi^2_{k_1,\ k_2}\varphi^2_{k_2,\ k_1}\,dxdy + 3\beta s_1 s_2^2 \iint\limits_{\Omega} \varphi_{k_1,\ k_2}\varphi^3_{k_2,\ k_1}\,dxdy + \\
&+ 3\beta s_1^2 \iint\limits_{\Omega} \varphi^2_{k_1,\ k_2}\varphi_{k_2,\ k_1}u^{\perp}\,dxdy + 3\beta s_1 \iint\limits_{\Omega} \varphi_{k_1,\ k_2}\varphi_{k_2,\ k_1}(u^{\perp})^2\,dxdy + \\
&+ 3\beta s_2^2 \iint\limits_{\Omega} \varphi^3_{k_2,\ k_1}u^{\perp}\,dxdy + 3\beta s_2 \iint\limits_{\Omega} \varphi^2_{k_2,\ k_1}(u^{\perp})^2\,dxdy = 0 \Big\}.
\end{aligned}$$

Approach 1. In the general case, identifying the number of solutions to the problem (2), (3), (5) causes difficulties. It is impossible to apply one general method. Let us consider one of the cases in which it is possible to identify the number of solutions to the problem being studied. Suppose there are pairs $\alpha_1, \beta_1 \in \mathbb{R}$ and $\alpha_2, \beta_2 \in \mathbb{R}$ such that upon substitution $v_1 = \alpha_1 s_1 + \beta_1 s_2$, $v_2 = \alpha_2 s_1 + \beta_2 s_2$, in the equations of the system (13), defining the set $\mathfrak{B}_2$, we can obtain a system of equations of the following form:

$$\begin{cases} a_1 v_1^3 + b_1 v_1^2 + c_1 v_1 + d_1 = 0, \\ a_2 v_2^3 + b_2 v_2^2 + c_2 v_2 + d_2 = 0. \end{cases} \tag{14}$$

Let us define for the first equation of the system (14) the Cardano formulas

$$\begin{aligned} Q_2^1(u) &= \left(\frac{3a_1c_1 - b_1^2}{9a_1^2}\right)^3 + \frac{1}{4}\left(\frac{2b_1^3}{27a_1^3} - \frac{b_1c_1}{3a_1^2} + \frac{d_1}{a_1}\right)^2, \\ R_2^1(u) &= v_1^2 \|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)} + 2v_1 \iint\limits_{\Omega} \varphi^3_{k_1,\ k_2} u^{\perp}\, dxdy + \iint\limits_{\Omega} \varphi^2_{k_1,\ k_2} (u^{\perp})^2\, dxdy, \end{aligned} \tag{15}$$

for the second equation of the system (14) of the Cardano formulas

$$\begin{aligned} Q_2^2(u) &= \left(\frac{3a_2c_2 - b_2^2}{9a_2^2}\right)^3 + \frac{1}{4}\left(\frac{2b_2^3}{27a_2^3} - \frac{b_2c_2}{3a_2^2} + \frac{d_2}{a_2}\right)^2, \\ R_2^2(u) &= v_2^2 \|\varphi_{k_2,\ k_1}\|^4_{L_4(\Omega)} + 2v_2 \iint\limits_{\Omega} \varphi^3_{k_2,\ k_1} u^{\perp}\, dxdy + \iint\limits_{\Omega} \varphi^2_{k_2,\ k_1} (u^{\perp})^2\, dxdy, \end{aligned} \tag{16}$$

and introduce the following sets

$$\begin{aligned} (\mathfrak{U}_2)^{\perp}_{+} &= \{u \in \mathfrak{U}^{\perp} : Q_2^1(u) > 0, Q_2^2(u) > 0\}, \\ (\mathfrak{U}_2)^{\perp}_{01} &= \{u \in \mathfrak{U}^{\perp} : R_2^1(u) = 0\}, \\ (\mathfrak{U}_2)^{\perp}_{02} &= \{u \in \mathfrak{U}^{\perp} : R_2^2(u) = 0\}, \\ (\mathfrak{U}_2)^{\perp}_{-} &= \{u \in \mathfrak{U}^{\perp} : Q_2^1(u) < 0, Q_2^2(u) < 0\}. \end{aligned}$$

Theorem 4. *Let $\alpha\beta < 0$ and $\mu = \lambda_{k_1,\ k_2}, k_1 \neq k_2$ and $\exists \alpha_1, \beta_1, \alpha_2, \beta_2$, such that (14). Then*
(i) for any $u_0 \in (\mathfrak{U}_2)^{\perp} \cap (\mathfrak{U}_2)^{\perp}_{-}$ there are nine solutions to the problem (2), (3), (5);
(ii) for any $u_0 \in (\mathfrak{U}_2)^{\perp} \cap (\mathfrak{U}_2)^{\perp}_{01} \cup (\mathfrak{U}_2)^{\perp}_{01}$ exists from two up to eight solutions to the problem (2), (3), (5);
(iii) for any $u_0 \in (\mathfrak{U}_2)^{\perp} \cap (\mathfrak{U}_2)^{\perp}_{+}$ there is a unique solution to the problem (2), (3), (5).

Proof. (i) Let's take a point $u_0 \in (\mathfrak{U}_2)^{\perp} \cap (\mathfrak{U}_2)^{\perp}_{-}$. According to Theorem 2, there are three solutions v_1^1, v_1^2, v_1^3 of the first equation from the system (14) and three solutions v_2^1, v_2^2, v_2^3 of the second equation from system (14), which means that point u_0 serves as the image of three points $u_0^1 = v_1^1 \varphi_{k_1,\ k_2} + v_2^1 \varphi_{k_2,\ k_1} + u^{\perp} \in \mathfrak{B}_1$, $u_0^2 = v_1^2 \varphi_{k_1,\ k_2} + v_2^2 \varphi_{k_2,\ k_1} + u^{\perp} \in \mathfrak{B}_1$, $u_0^3 = v_1^3 \varphi_{k_1,\ k_2} + v_2^3 \varphi_{k_2,\ k_1} + u^{\perp}$. But when reversely replaced by s_1 and s_2, the point u_0 should serve as the image of nine points. According to the Theorem on the existence of a solution to the Showalter–Sidorov problem [22], the problem (2), (3), (5) should have nine different solutions.

(ii) Let's take the point $u_0 \in (\mathfrak{U}_2)^{\perp} \cap (\mathfrak{U}_2)^{\perp}_{01} \cup (\mathfrak{U}_2)^{\perp}_{01}$. According to Theorem 2, there are from two to eight (s_1, s_2) solutions to the system of equations (14), which means that the point u_0 serves as the image of two to eight points belonging to $\mathfrak{B}_2$.

(iii) Let's take the point $u_0 \in \mathfrak{U}^{\perp} \cap \mathfrak{U}^{\perp}_{+}$. According to Theorem 2, there is one solution v_1 and v_2 to the equation from the system (14), which means that the point u_0 serves as the image of only one point $u_0 = v_1\varphi_{k_1,\,k_2} + v_2\varphi_{k_2,\,k_1} + u^{\perp} \in B_2$ or after reverse substitution $u_0 = s_1\varphi_{k_1,\,k_2} + s_2\varphi_{k_2,\,k_1} + u^{\perp} \in B_2$. According to the Theorem on the existence of a solution to the Showalter–Sidorov problem, the problem (2), (3), (5) should have only one solution.

□

Approach 2. The approach described above cannot always be applied in a general form, so let's consider another approach that allows us to investigate the problem (2), (3), (5) for the existence of non-unique solutions. To do this, let's solve the first equation of the system (13), which defines the set $\mathfrak{B}_2$, with respect to one of the variables, for example s_1. Thus, solving the cubic equation, we obtain up to three solutions $s_{1k}(s_2)$. Substituting $s_{1k}(s_2)$ into the second equation of the system (13) we obtain up to three third degree equations depending on s_2. Thus, depending on the values of the parameters of the equation (2), there can be from one to nine solutions to the problem (2), (3), (5).

2. Numerical Experiment

Let us consider more specific examples of numerical investigation of the non-uniqueness of solutions to the Showalter–Sidorov problem (2), (3), (5) using approach 1 and 2 described in paragraph 1.

Example 2. Let $\Omega = (0,\pi) \times (0,\pi)$, consider the Showalter–Sidorov–Dirichlet problem:

$$\begin{gathered} 5\,(u(x,y,0) - u_0(x,y)) + (u_{xx}(x,y,0) + u_{yy}(x,y,0) - u_0(x,y)) = 0, \\ x \in (0,\pi), y \in (0,\pi), \end{gathered} \tag{17}$$

$$\begin{gathered} u(0,y,t) = u(\pi,y,t) = 0,\ y \in (0,\pi),\ t \in (0,1), \\ u(x,0,t) = u(x,\pi,t) = 0,\ x \in (0,\pi),\ t \in (0,1), \end{gathered} \tag{18}$$

for the Hoff equation

$$5u_t + u_{xxt} + u_{yyt} = \alpha u + \beta u^3, \tag{19}$$

where $u_0(x,y) = \frac{2}{\pi}\sin x \sin y - \frac{2}{\pi}\sin 2x \sin 2y$. It is required to identify the existence and number of solutions to the problem (17) – (19).

Under the conditions of this example, μ coincides with the second eigenvalue $\lambda_{1,2} = 5$ of the homogeneous Dirichlet problem for the operator $(-\Delta)$ and the considered eigenvalue $\lambda_{1,2}$ corresponds to two eigenfunctions $\varphi_{1,2}(x,y) = \dfrac{2}{\pi}\sin x \sin 2y$ and $\varphi_{2,1}(x,y) = \dfrac{2}{\pi}\sin 2x \sin y$. Following the algorithm, we represent the function $u(x,y,0)$ in the form $u(x,y,0) = s_1\varphi_{1,2} + s_2\varphi_{2,1} + u^{\perp}$, $u^{\perp} \in \mathfrak{U}^{\perp}$, the set $\mathfrak{B}_2$ take the form:

$$\mathfrak{B}_2 = \left\{ (s_1, s_2, u^{\perp}) \in \mathbb{R}^2 \times L_4(\Omega) : \left\{ \begin{array}{c} \alpha s_1 + \frac{9\beta s_1}{\pi^2} + \frac{6\beta s_2}{\pi^2} + \frac{9\beta s_1^3}{4\pi^2} + \frac{3\beta s_1 s_2^2}{\pi^2} = 0, \\ \alpha s_2 + \frac{6\beta s_1}{\pi^2} + \frac{9\beta s_2}{\pi^2} + \frac{9\beta s_2^3}{4\pi^2} + \frac{3\beta s_1^2 s_2}{\pi^2} = 0 \end{array} \right. \right\}. \tag{20}$$

Substituting $v_1 = s_1 - 2s_2, v_2 = s_1 - s_2$ into (20), we obtain a system of cubic equations:

$$\begin{cases} -\dfrac{7\beta v_1^3 + 12\pi^2\alpha v_1 + 36\beta v_1}{36\pi^2} = 0, \\ \\ \dfrac{21\beta v_2^3 + 16\pi^2\alpha v_2 + 48\beta v_2}{36\pi^2} = 0, \end{cases} \quad (21)$$

equivalently defining the set $\mathfrak{B}_2$. Let us define for the first equation in (21)

$$Q_2^1(u) = \left(\frac{252\pi^2\alpha\beta + 756\beta^2}{441\beta^2}\right)^3,$$

and for the second equation in (21)

$$Q_2^2(u) = \left(\frac{1008\pi^2\alpha\beta + 3024\beta^2}{3969\beta^2}\right)^3.$$

Let's consider the special case when $\alpha = 1$, and $\beta = -3,5$, then $Q_2^1 = 0,00109023 > 0$, $Q_2^2 = 0,00009571 > 0$, from which it follows that the solution to the Showalter–Sidorov–Dirichlet problem (17) – (19) is unique. To construct a numerical solution to the problem (17) – (19), we use an algorithm for finding an approximate solution to the Showalter–Sidorov problem based on the modified Galerkin method. Following Galerkin's method, we look for an approximate solution to the problem under consideration in the form of the sum

$$u_m(x,y,t) = \sum_{k_1=1}^{m}\sum_{k_2=1}^{m} u_{k_1,k_2}(t)\varphi_{k_1,k_2}(x,y).$$

Let us briefly present the numerical solution algorithm; a detailed description of the algorithm is presented in the work [18]:

Step 1. Construct approximate sums and substitute them into the equation.

Step 2. Multiply the resulting equation scalarly in $L_2(\Omega)$ by the eigenfunctions $\varphi_{k_1,k_2}(x,y)$ and obtain a system of equations for the unknowns $u_{k_1,k_2}(t)$.

Step 3. Depending on the parameter μ, we obtain differential or algebraic equations in this system. In this example, a system of algebraic-differential equations is obtained due to the degeneracy of the equation. We find m or $(m-1)$ or $(m-2)$ initial conditions depending on the number of algebraic equations of the system.

Step 4. We numerically solve the system of algebraic-differential equations with m or $(m-1)$ or $(m-2)$ initial conditions and find the unknown functional coefficients $u_{k_1,k_2}(t)$ in the approximate solution $u_m(x,y,t)$.

Step 5. We build a graph of the approximate solution.

The numerical solution of the problem (17) – (19) in the case of $\alpha = 1$, $\beta = -3,5$ is presented in Fig. 1.

In the case when $\alpha = 1$, $\beta = -0,1$, then $Q_2^1 = -16,3518 < 0$, $Q_2^2 = -14,355 < 0$, which implies the existence nine solutions to the Showalter–Sidorov–Dirichlet problem (17) – (19). The resulting numerical solution to the problem based on the presented algorithm is shown in Fig. 2.

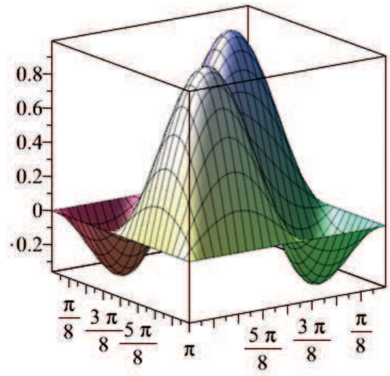

Fig. 1. Numerical solution of the $u_m(x,y,t)$ problem (17) – (19)

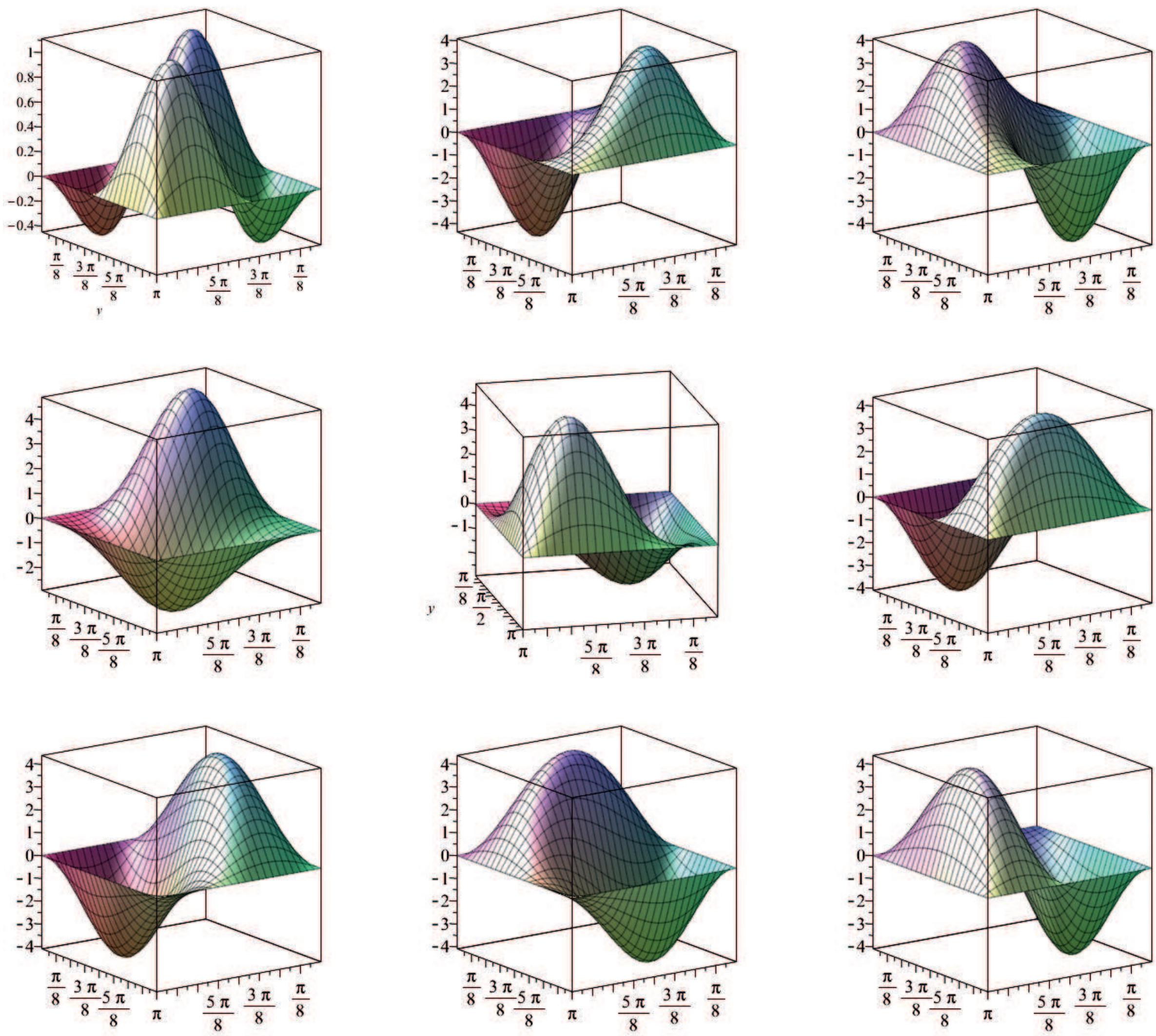

Fig. 2. Numerical solution $u(x,y,t)$ of the problem (17) – (19) in the case of non-unique solution

Example 3. Let's consider finding solutions to the problem (17) – (19) based on the second approach. To do this, we transform the equations of the system (13), adding them, and get

$$\begin{cases} \alpha s_1 + \alpha s_2 + \dfrac{9\beta s_1}{\pi^2} + \dfrac{6\beta s_2}{\pi^2} + \dfrac{9\beta s_1^3}{4\pi^2} + \dfrac{3\beta s_1 s_2^2}{\pi^2} + \\ + \dfrac{6\beta s_1}{\pi^2} + \dfrac{9\beta s_2}{\pi^2} + \dfrac{9\beta s_2^3}{4\pi^2} + \dfrac{3\beta s_1^2 s_2}{\pi^2} = 0, \\ \\ \alpha s_2 + \dfrac{6\beta s_1}{\pi^2} + \dfrac{9\beta s_2}{\pi^2} + \dfrac{9\beta s_2^3}{4\pi^2} + \dfrac{3\beta s_1^2 s_2}{\pi^2} = 0. \end{cases} \tag{22}$$

Let us decompose the first equation of the system (22) into two factors:

$$(s_1 + s_2)\left(\frac{(9s_1^2 + 3s_1 s_2 + 9s_2^2 + 60)\beta + 4\pi^2\alpha}{4\pi^2}\right) = 0. \tag{23}$$

Let us solve the equation (23) with respect to the variable s_1. Thus, we get up to three solutions $s_{1k}(s_2), k = 1, \ldots, 3$. Substituting $s_{1k}(s_2)$ into the second equation of the system (22) we obtain up to three third degree equations depending on s_2. With the reverse substitution of s_2 into $s_{1k}(s_2)$ and depending on the values of the parameters of the equation (19), there can be from one to nine solutions to the problem (17) – (19).

Let's consider the special case when $\alpha = 1$, $\beta = -0,5$, in this case there are five solutions to the Showalter–Sidorov–Dirichlet problem (17) – (19). In Fig. 3 shows the resulting numerical solution to the problem based on the algorithm presented in example 2.

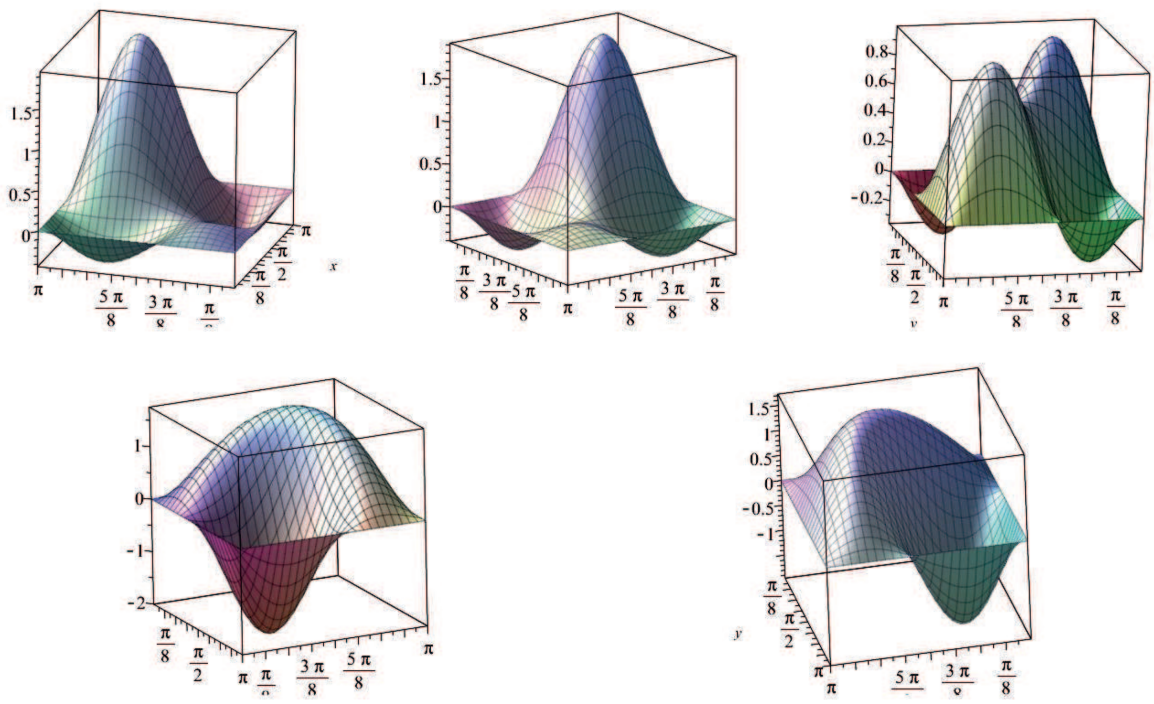

Fig. 3. Numerical solution of $u(x, y, t)$ problem (17) – (19)

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ИССЛЕДОВАНИЕ ЕДИНСТВЕННОСТИ РЕШЕНИЯ ЗАДАЧИ ШОУОЛТЕРА – СИДОРОВА ДЛЯ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ХОФФА. МОРФОЛОГИЯ ФАЗОВОГО ПРОСТРАНСТВА

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Статья посвящена изучению морфологии фазового пространства математической модели деформации двутавровой балки, которое лежит на гладких банаховых многообразиях с особенностями (k-сборка Уитни) в зависимости от параметров задачи. Математическая модель изучена в случае, когда оператор при производной по времени является вырожденным. Исследование вопроса неединственности решения задачи Шоуолтера – Сидорова для модели Хоффа в двумерной области проведено на основе метода фазового пространства, который был разработан Г.А. Свиридюком. Найдены условия неединственности решения в случае, когда размерность ядра оператора при производной по времени равна 1 или 2. Представлены два подхода для выявления количества решений задачи Шоуолтера – Сидорова в случае, размерности ядра оператора при производной по времени равного 2. Приведены примеры, иллюстрирующие неединственность решения исследуемой задачи на прямоугольнике.

Ключевые слова: уравнения соболевского типа; задача Шоуолтера – Сидорова; метод фазового пространства; сборка Уитни; уравнение Хоффа; неединственность решений.

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