КРАТКИЕ СООБЩЕНИЯ

MSC 35R30

AN INVERSE PROBLEM FOR A LINEARIZED QUASI-STATIONARY PHASE FIELD MODEL WITH DEGENERACY

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The inverse problem for a linearized quasi-stationary phase field model is considered. The inverse problem is reduced to a linear inverse problem for the first order differential equation in a Banach space with a degenerate operator at the derivative and an overdetermination condition on the degeneracy subspace. The unknown parameter in the problem dependens on the source time function. The theorem of existence and uniqueness of classical solutions is proved by methods of degenerate operator semigroup theory at some additional conditions on the operator. General results are applied to the original inverse problem.

Keywords: inverse problem, phase field model, Sobolev type equation, degenerate operator, operator semigroup, Banach spaces.

Preface

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary $\partial\Omega$ of C^∞ class, $T > 0$, β, $\delta \in \mathbb{R}$. Consider the initial-boundary value problem

$$(\beta + \Delta)(v(x,0) - v_0(x)) = 0, \quad x \in \Omega, \tag{1}$$

$$(1-\delta)v + \delta\frac{\partial v}{\partial n}(x,t) = (1-\delta)w + \delta\frac{\partial w}{\partial n}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{2}$$

for the system of equations

$$v_t(x,t) = \Delta v(x,t) - \Delta w(x,t) + b_1(x,t)u(t), \quad (x,t) \in \Omega \times [0,T], \tag{3}$$

$$0 = v + (\beta + \Delta)w + b_2(x,t)u(t), \quad (x,t) \in \Omega \times [0,T], \tag{4}$$

with overdetermination condition on the subspace of degeneracy

$$\int\limits_\Omega K(y)w(y,t)dy = \psi(t), \quad (x,t) \in \Omega \times [0,T]. \tag{5}$$

Up to a linear change of functions $v(x,t)$, $w(x,t)$, the system coincides with the linearization of the quasistationary phase-field model, describing phase transitions of the first kind in terms of the mesoscopic theory. The unknown functions of the inverse problem (1)–(5) are $v(x,t)$, $w(x,t)$, $u(t)$. The problem is investigated within the framework of a linear inverse problem for an abstract differential equation with a degenerate operator at the derivative, i.e. the Sobolev type equation.

Linear inverse problems for the Sobolev type equations were studied in [1, 2] with an unknown time-independent element u. Problems with an unknown time-dependent element u were considered in linear case in [3], and in nonlinear case in [4, 5]. However, an overdetermination operator, herewith, acted on the resolving Sobolev type equations semigroup image. In the present paper this operator acts on the semigroup kernel.

1. Statement of the abstract problem

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$ be Banach spaces. Consider operators $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ (i. e. linear and continuous) with $\ker L \neq \{0\}$, $M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$ (linear, closed and densely defined), $\Phi \in \mathcal{L}(\mathcal{X};\mathcal{U})$, $B \in C^1([0,T];\mathcal{L}(\mathcal{U};\mathcal{Y}))$, functions $y \in C^1([0,T];\mathcal{Y})$, $\Psi \in C^1([0,T];\mathcal{U})$ and an element $x_0 \in D_M$. Here D_M is a domain of the operator M, endowed with the graph norm $\|x_0\|_{D_M} = \|x_0\|_{\mathcal{X}} + \|Mx_0\|_{\mathcal{Y}}$.

Theorem 1. [6] *Let $p \in \{0\} \cup \mathbb{N}$, the operator M be strongly (L,p)-radial. Then*

(i) $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$;

(ii) *a projector along $\mathcal{X}^0$ on $\mathcal{X}^1$ (along $\mathcal{Y}^0$ on $\mathcal{Y}^1$) has the form*

$$P = s\text{-}\lim_{\mu \to +\infty} (\mu R^L_\mu(M))^{p+1}, \quad (Q = s\text{-}\lim_{\mu \to +\infty} (\mu L^L_\mu(M))^{p+1});$$

(iii) *$QL = LP$, $QMx = MPx$ for all $x \in D_M$;*

(iv) *$L_k \equiv L|_{\mathcal{X}^k} \in \mathcal{L}(\mathcal{X}^k;\mathcal{Y}^k)$, $M_k \equiv M|_{D_M \cap \mathcal{X}^k} \in \mathcal{C}l(\mathcal{X}^k;\mathcal{Y}^k)$, $k = 0,1$;*

(v) *operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$ exist;*

(vi) *the operator $H = M_0^{-1}L_0$ is nilpotent of a degree not greater, than p;*

(vii) *there is a strongly continuous operators semigroup $\{V(t) \in \mathcal{L}(\mathcal{X}) : t \geq 0\}$, that resolves the equation $L\dot{x}(t) = Mx(t)$.*

While the operator M is strongly (L,p)-radial, let us consider the inverse problem

$$L\dot{x}(t) = Mx(t) + Bu(t) + y(t), \quad t \in [0,T], \tag{6}$$

$$Px(0) = x_0, \tag{7}$$

$$\Phi x(t) = \Psi(t), \quad t \in [0,T], \tag{8}$$

which is to find a pair of functions $x \in C^1([0,T];\mathcal{X}) \cap C([0,T];D_M)$ and $u \in C^1([0,T];\mathcal{U})$, named as a solution.

Theorem 2. *Let the operator M be strongly (L,p)-radial, $\Phi \in \mathcal{L}(\mathcal{X};\mathcal{U})$, $\Phi H = \mathbb{O}$, $\mathcal{X}^1 \subset \ker \Phi$, $B \in C^1([0,T];\mathcal{L}(\mathcal{U};\mathcal{Y}))$, $y \in C^1([0,T];\mathcal{Y})$, $(I-Q)y \in C^{p+1}([0,T];\mathcal{Y})$, $\Psi \in C^1([0,T];\mathcal{U})$, the inverse operator $(\Phi M_0^{-1}(I-Q)B(t))^{-1}$ exists for all $t \in [0,T]$ and $(\Phi M_0^{-1}(I-Q)B)^{-1} \in C^1([0,T];\mathcal{L}(\mathcal{U}))$, $x_0 \in D_M \cap \mathcal{X}^1$. Then there is a unique solution $(x;u)$ of the problem (6)–(8). It has the form*

$$x(t) = V(t)x_0 + \int_0^t V(t-s)L_1^{-1}Q(B(s)u(s) + y(s))ds -$$

$$-M_0^{-1}(I-Q)B(t)u(t) - \sum_{k=0}^{p} H^k M_0^{-1} \left((I-Q)y(t)\right)^{(k)}, \tag{9}$$

$$u(t) = -(\Phi M_0^{-1}(I-Q)B(t))^{-1} \left(\Psi(t) + \sum_{k=0}^{p} \Phi H^k M_0^{-1}((I-Q)y)^{(k)}(t) \right) \tag{10}$$

and satisfies the following conditions:

$$\|x\|_{C^1([0,T];\mathcal{X})} \leq c \left(\|Px_0\|_{D_M} + \|\Psi\|_{C^1([0,T];\mathcal{U})} + \|y\|_{C^{p+1}([0,T];\mathcal{Y})} \right), \tag{11}$$

$$\|u\|_{C^1([0,T];\mathcal{U})} \leq c \left(\|\Psi\|_{C^1([0,T];\mathcal{U})} + \|y\|_{C^{p+1}([0,T];\mathcal{Y})} \right), \tag{12}$$

where a constant $c > 0$ does not depend on x_0, y, Ψ.

Proof. Act with the operator Φ on the solution x of the direct problem (6), (7) with the known element u. Then

$$\Phi x(t) = \Phi(I-P)x(t) = -\Phi M_0^{-1}(I-Q)B(t)u(t) - \sum_{k=0}^{p} \Phi H^k M_0^{-1}((I-Q)y)^{(k)}(t) = \Psi(t) \quad (13)$$

for all $t \in [0,T]$ due to the overdetermination condition (8), the conditions $\mathcal{X}^1 \subset \ker \Phi$, $\Phi H = \mathbb{O}$ and the form of x (see [6, 7]). Thus, the formula (10) holds.

The formula (9) is obtained in [6], when u is unknown. At the same time the equality (13) is taken into account. Estimates (11), (12) follow from (9), (10) and an operator semigroup $\{V(t) \in \mathcal{L}(\mathcal{X}) : t \geq 0\}$ exponential growth.

□

2. Inverse problem for a linearized quasi-stationary phase field model

Reduce the problem (1)–(5) to (6)–(8). To do this,let us assume $\mathcal{X} = \mathcal{Y} = (L_2(\Omega))^2$, $\mathcal{U} = \mathbb{R}$,

$$L = \begin{pmatrix} I & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad M = \begin{pmatrix} \Delta & -\Delta \\ I & \beta I + \Delta \end{pmatrix}, \quad B(t) = \begin{pmatrix} b_1(\cdot,t) \\ b_2(\cdot,t) \end{pmatrix}, \quad \Psi(t) = \psi(t),$$

$$H_\delta^2(\Omega) = \left\{ h \in H^2(\Omega) : \left(\delta \frac{\partial}{\partial n} + (1-\delta) \right) h(x) = 0,\ x \in \partial\Omega \right\},$$

$D_M = (H_\delta^2(\Omega))^2$. Thereby, $L \in \mathcal{L}(\mathcal{X})$, $M \in \mathcal{C}l(\mathcal{X})$, $\ker L \neq \{0\}$.

Denote $Aw = \Delta w$, $D_A = H_\delta^2(\Omega) \subset L_2(\Omega)$. Let $\{\varphi_k : k \in \mathbb{N}\}$ be orthonormal (in the sense of the scalar product $\langle \cdot, \cdot \rangle$ in $L_2(\Omega)$) eigenfunctions of the operator A, numbered in decreasing eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$, counting multiplicities. Let $-\beta \in \sigma(A)$, define $\delta_k = (\beta + \lambda_k)^{-1}(\beta + 1 + \lambda_k)\lambda_k$, for $\lambda_k \neq -\beta$. Using the expansion in the basis $\{\varphi_k : k \in \mathbb{N}\}$ in a space $L_2(\Omega)$, determine operators

$$(R_\mu^L(M))^2 = \begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{(\mu - \delta_k)^2} & \mathbb{O} \\ \sum\limits_{\lambda_k \neq -\beta} \frac{-\langle \cdot, \varphi_k \rangle \varphi_k}{(\beta + \lambda_k)(\mu - \delta_k)^2} & \mathbb{O} \end{pmatrix},$$

$$(L_\mu^L(M))^2 = \begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{(\mu - \delta_k)^2} & \sum\limits_{\lambda_k \neq -\beta} \frac{\lambda_k \langle \cdot, \varphi_k \rangle \varphi_k}{(\beta + \lambda_k)(\mu - \delta_k)^2} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

Hence, considering the Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$, we obtain the strong $(L,1)$-radiality of the operator M [6]. By formulas $P = s\text{-}\lim\limits_{\mu \to +\infty} (\mu R_\mu^L(M))^2$, $Q = s\text{-}\lim\limits_{\mu \to +\infty} (\mu L_\mu^L(M))^2$ we derive the projectors

$$P = \begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \langle \cdot, \varphi_k \rangle \varphi_k & \mathbb{O} \\ -\sum\limits_{\lambda_k \neq -\beta} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{\beta + \lambda_k} & \mathbb{O} \end{pmatrix}, \quad Q = \begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \langle \cdot, \varphi_k \rangle \varphi_k & \sum\limits_{\lambda_k \neq -\beta} \frac{\lambda_k \langle \cdot, \varphi_k \rangle \varphi_k}{\beta + \lambda_k} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

Theorem 3. *Let $-\beta \in \sigma(A)$, $K \in L_2(\Omega)$, $\langle K, \varphi_k \rangle = 0$ for $\lambda_k = -\beta$, $b_i \in C^1([0,T]; L_2(\Omega))$, $i = 1,2$, and $\langle b_1(\cdot,t), \varphi_k \rangle = 0$ for $\lambda_k \neq -\beta$, $\langle K, b_2(\cdot,t) \rangle \neq 0$ for all $t \in [0,T]$, $\psi \in C^1[0,T]$, $v_0 \in H_\delta^2(\Omega)$. Then there exists a unique solution of the problem* (1)–(5).

Proof. To prove this theorem it is sufficient to verify the conditions of Theorem 2.

□

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ОБРАТНАЯ ЗАДАЧА ДЛЯ ЛИНЕАРИЗОВАННОЙ КВАЗИСТАЦИОНАРНОЙ МОДЕЛИ ФАЗОВОГО ПОЛЯ С ВЫРОЖДЕНИЕМ

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Рассмотрена обратная задача для линеаризованной квазистационарной модели фазового поля. Она редуцирована к линейной обратной задаче для дифференциального уравнения первого порядка в банаховом пространстве с вырожденным оператором при производной и с переопределением на подпространстве вырождения. Неизвестный параметр в задаче представляет собой зависящую от времени функцию источника. При некоторых дополнительных условиях на оператор переопределения методами теории вырожденных полугрупп операторов доказана теорема существования и единственности классического решения. Общий результат использован при исследовании исходной обратной задачи.

Ключевые слова: обратная задача, модель фазового поля, уравнение соболевского типа, вырожденный оператор, полугруппы операторов, банаховы пространства.

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