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ON FACTORIZATION OF A DIFFERENTIAL OPERATOR ARISING IN FLUID DYNAMICS

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Spectral properties of linear operators are very important in stability analysis of dynamical systems. The paper studies the non-selfadjoint second order differential operator that originated from a steady state stability problem in dynamic of viscous Newtonian fluid on the inner surface of horizontally rotating cylinder in the presence of gravitational field. The linearization of the thin liquid film flow in the lubrication limit about the uniform coating steady state results into the operator which domain couples two subspaces spanned by positive and negative Fourier exponents which are not invariant subspaces of the operator. We prove that the operator admits factorization and use this new representation of the operator to prove compactness of its resolvent and to find its domain.

Keywords: factorization, lubrication approximation, fluid mechanics, forward-backward heat equation.

Introduction

Depending on the parameters involved, the dynamics of the film of viscous fluid can be described by different asymptotic equations. Under assumption that the film is thin enough for viscous entrainment to compete with gravity, the time evolution model of a thin film of liquid on the inner surface of a cylinder rotating in a gravitational field was based on the lubrication approximation and examined by Benilov, O'Brien, and Sazonov [2, 3]. The related Cauchy problem has the following form:

$$y_t + l[y] = 0, \quad y(0,x) = y_0, \quad y(-\pi,t) = y(\pi,t), \quad x \in [-\pi,\pi], \quad t > 0 \tag{0.1}$$

where

$$l[y] = \frac{d}{d\,x}\left((1 - a\,\cos x)y(x) + b\,\sin x \cdot \frac{d\,y(x)}{d\,x}\right), \quad a, b > 0 \tag{0.2}$$

Eigenmode solutions are very important in stability analysis, because even a single growing mode can destabilize an otherwise stable system. In case when all modes are bounded in time and the corresponding eigenfunctions form a complete set, the system normally regarded as a stable one. Because, an arbitrary initial condition can be represented as a series of these eigenmodes; and since all of them are stable, so expected to be the solution to the initial- value problem.

There are however counterexamples to the arguments above when each term of the series is bounded but the series as a whole diverges and the solution develops a singularity in a finite time. This effect was observed by Benilov, O'Brien, and Sazonov [2] for the problem 0.1 when parameter in (0.2) $a = 0$. For this case when the effect of gravitational drainage was neglected because of infinitesimally thin film they studied stability of the problem asymptotically and numerically. It was shown that even for infinitely smooth initial values numerical solutions blow up after a small number of iterations.

The spectrum of the linear operator L that is defined by the operation $l[.]$ and periodic boundary conditions $y(-\pi) = y(\pi)$ for the special case when the parameter $a = 0$ was studied rigorously in [8, 6, 9]. Using different approaches they justified that if the parameter b restricted to the interval $[0, 2]$ then the operator L is well defined in the sense that it admits closure in $L^2(-\pi, \pi)$ with non-empty resolvent set without breaking the boundary conditions $y(-\pi) = y(\pi)$. The spectrum of the operator L is discrete and consists of simple pure imaginary eigenvalues only. As a result all eigenfunctions have the following symmetry $y_\lambda(-x) = \overline{y_\lambda(x)}$. The more general operator with the function $sin(x)$ replaced by the arbitrary 2π-periodic functions was studied in [4] and it was proved that this operator multiplied by i belongs to a wide class of PT-symmetric operators which are not similar to self-adjoint but nevertheless possesses purely real spectrum due to some obvious and hidden symmetries.

The phenomenon of the coexistence of the neutrally stable modes with explosive instability of the numerical solutions [2] (which correspond to drops of fluid forming on the ceiling of the cylinder where the effect of the gravity is the strongest) was studied analytically and explained in terms of the absence of the Riesz basis property of the set of eigenfunctions in [5]. The question of a conditional basis property of the set of eigenfunction is still open.

For the case when $a \neq 0$, as it was discussed in [3], the spectral properties of the operator L are not expected to differ a lot from the case $a = 0$.

The goal of this paper is to find a factorization of the operator L (under some restrictions on parameters a and b) that would be in some sense similar to one we constructed for the special case $a = 0$ in [7] (in this case the operator L is J-self-adjoint with the operator J defined as a shift $J(f(x)) = f(\pi - x)$) and to examine some properties of the operator L using this factorization. The main difficulty to overcome here is an existing coupling between two subspaces spanned by positive and negative Fourier exponents which are not invariant subspaces of the operator L if $a \neq 0$. We also prove that the non-self-adjoint differential operator L has compact resolvent and as result spectrum of L is discrete with the only accumulating point at infinity.

1. Factorization of the non-self-adjoint operator L

We denote by $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ the domain and the range of linear operator T respectively. The notation $\mathcal{L}^2$ is used for the standard Lebesgue space of scalar functions defined on the interval $(-\pi, \pi)$. From here on L is the indefinite convection-diffusion operator L :

$$\mathcal{L}^2 \mapsto \mathcal{L}^2, \quad (Ly)(x) = \frac{d}{d\,x}\left((1 - a\,\cos x)y(x) + b\,\sin x \cdot \frac{d\,y(x)}{d\,x}\right), \quad b \neq 0$$

with the domain of all absolutely continuous 2π-periodic functions $y(x)$ such that $(Ly)(x) \in \mathcal{L}^2$. The latter means, in particular, that we consider $y(x)$ such that the function

$$\left((1 - a\,\cos x)y(x) + b\,\sin x \cdot \frac{d\,y(x)}{d\,x}\right)$$

can be defined at zero as a continuous function and by this way it is converted in an absolutely continuous function.

In addition, we define the operator S :

$$\mathcal{L}^2 \mapsto \mathcal{L}^2, \quad (Sy)(x) = y'(x),$$

where $y'(x) \in \mathcal{L}^2$, $y(-\pi) = y(\pi)$, and the operator $M : \mathcal{L}^2 \mapsto \mathcal{L}^2$,

$$(My)(x)\colon = (1 - a\,\cos x)y(x) + b\,\sin x \cdot \frac{d\,y(x)}{d\,x}$$

with the domain of all functions $y(x) \in \mathcal{L}^2$ absolutely continuous on $(-\pi,\,0) \cup (0,\,\pi)$ and such that $(My)(x) \in \mathcal{L}^2$. Note, that, for example, $y(x) = x^{-1/3} \in \mathfrak{D}(M)$. The operator M can also be represented by the following expression

$$(My)(x) = \big(1 - (a+b)\,\cos x)y(x) + b\,(\sin x \cdot y(x)\big)'.$$

Theorem 1. *If the parameters a and b satisfy the inequality $2a + b < 2$, then L is a closed operator with a closed range and $L = SM$.*

Proof. Let us consider the operator A :

$$\mathcal{L}^2 \mapsto \mathcal{L}^2, \quad (Ay)(x) = (\sin(x)y(x))'$$

with $\mathfrak{D}(A) = \{y(x) \mid y(x), (Ay)(x) \in \mathcal{L}^2\}$. Then $\mathfrak{D}(A) = \mathfrak{D}(M)$ and a function $y(x) \in \mathfrak{D}(A)$ can be written as

$$y(x) = \frac{1}{\sin(x)} \cdot \Big(\big(c(1 + \operatorname{Sign} x)/2 + c_1(1 - \operatorname{Sign} x)/2\big) + \int_0^x \theta(t)dt\Big), \quad \theta(t) \in \mathcal{L}^2. \qquad (1.1)$$

If $x > 0$ then

$$|\int_0^x \theta(t)dt| \le \alpha(x) \cdot x^{1/2},$$

where $\alpha(x) = \big(\int_0^x |\theta(x)|^2 dx\big)^{1/2}$. Since the two summands in (1.1) have different growth orders as $x \to 0$ this implies that if $y(x) \in \mathcal{L}^2$ then $c = 0$ and

$$y(x) = \frac{1}{\sin(x)} \cdot \int_0^x \theta(t)dt. \qquad (1.2)$$

Moreover,

$$|y(x)| \le \frac{x^{1/2}}{\sin(x)} \cdot \alpha(x). \qquad (1.3)$$

A small modification of the same reasoning leads to the following estimation for every $x \in (-\pi, \pi)$

$$|y(x)| \le \frac{|x|^{1/2}}{|\sin(x)|} \cdot \alpha(x). \qquad (1.4)$$

with $\alpha(x) = \big|\int_0^x |\theta(x)|^2 dx\big|^{1/2}$.

Alternatively the same function $y(x)$ can be written as

$$y(x) = \frac{1}{\sin(x)} \cdot \big(\tilde{c} - \int_x^\pi \theta(t)dt\big), \quad \theta(t) \in \mathcal{L}^2. \qquad (1.5)$$

with the same $\theta(x)$ as in (1.1). Representation (1.5) yields the following relations

$$y(x) = \frac{-1}{\sin(x)} \cdot \int_x^\pi \theta(t)dt \qquad (1.6)$$

and

$$|y(x)| \le \frac{(\pi - x)^{1/2}}{\sin(x)} \cdot \beta(x) \qquad (1.7)$$

with $\beta(x) = \big|\int_x^\pi |\theta(x)|^2 dx\big|^{1/2}$.

It follows from (1.2) and (1.6) that

$$\int_0^\pi \theta(t)dt = 0. \qquad (1.8)$$

Starting from the point $-\pi$ one can also obtain that

$$y(x) = \frac{1}{\sin(x)} \cdot \int_{-\pi}^{x} \theta(t)dt, \tag{1.9}$$

$$|y(x)| \leq \frac{(-\pi + x)^{1/2}}{|\sin(x)|} \cdot \gamma(x) \tag{1.10}$$

with $\gamma(x) = \big(\int_{-\pi}^{x} |\theta(x)|^2 dx \big|^{1/2}$ and

$$\int_{-\pi}^{0} \theta(t)dt = 0. \tag{1.11}$$

Now we are ready to calculate M^*. Using smooth functions $y(x)$ such that $y(x) \equiv 0$ in some neighborhoods of the points $-\pi$, 0 and π (neighborhoods depend of $y(x)$) it is easy to show that

$$(M^* z)(x) = \big(1 - a\, \cos x\big) z(x) - b\, \big(\sin x \cdot z(x)\big)'$$

for every $z(x) \in \mathfrak{D}(M^*)$. Since the condition $z(x) \in \mathfrak{D}(M^*)$ yields

$$z(x) \in \mathcal{L}^2 \;\text{ and }\; (M^* z)(x) \in \mathcal{L}^2, \tag{1.12}$$

$z(x) \in \mathfrak{D}(A)$ and for $z(x)$ the conditions of the type (1.4), (1.7) and (1.10) are satisfied. So,

$$\lim_{x \to 0} y(x)z(x) \sin x = \lim_{x \to -\pi+0} y(x)z(x) \sin x = \lim_{x \to \pi-0} y(x)z(x) \sin x = 0\ .$$

Taking into account the latter one can check that

$$(My, z) = (y, M^{\#} z)$$

for every $y(x), z(x) \in \mathfrak{D}(M) = \mathfrak{D}(A)$, where $\mathfrak{D}(M^{\#}) = \mathfrak{D}(A)$ and $M^{\#}$ is defined by the same differential expression as M^*. Thus, $\mathfrak{D}(M) = \mathfrak{D}(M^*)$. The same reasoning shows that $M^{**} = M$, so M is closed.

Let

$$(1 - a\, \cos x)y(x) + b\, \sin x \cdot \frac{d\, y(x)}{d\, x} = u(x), \quad y(x), u(x) \in \mathcal{L}^2.$$

Our aim is to express $y(x)$ via $u(x)$. Let $x \in (-\pi,\, \pi)$, $x \neq 0$. Then $y(x) =$

$$\big(c(1 + \mathrm{Sign}\, x)/2 + c_1(1 - \mathrm{Sign}\, x)/2\big) \cdot (\sin|x|)^{a/b} \cdot (\cot|x|/2)^{1/b} +$$

$$\frac{1}{b}(\sin|x|)^{a/b} \cdot (\cot|x|/2)^{1/b} \int_0^x u(t)(\sin|t|)^{-\frac{a}{b}} \cdot (\sin t)^{-1} \cdot (\tan|t|/2)^{1/b} dt,$$

where c and c_1 are constants. The estimations that follow closely depend of a relation between a and b. We assume that $2a + b < 2$. Then for $x > 0$

$$\int_0^x u(t)(\sin t)^{-(\frac{a}{b}+1)} \cdot (\tan t/2)^{1/b} dt = \int_0^x v(t)(t)^{-(\frac{a}{b}+1)} \cdot (t/)^{1/b} dt,$$

where $v(t) = u(t)(\sin t)^{-(\frac{a}{b}+1)} \cdot (\tan t/2)^{1/b} (t)^{(\frac{a}{b}+1)} \cdot (t)^{-1/b}$, so,

$$|\int_0^x u(t)(\sin t)^{-(\frac{a}{b}+1)} \cdot (\tan t/2)^{1/b} dt| \leq$$

$$\sqrt{\frac{b}{2 - 2a - b}} \cdot x^{\frac{2-2a-b}{2b}} \cdot \big(\int_0^x |v(t)|^2 dt\big)^{1/2}.$$

Thus, the first summand (if $c \neq 0$) for $y(x)$ has the order $x^{\frac{a-1}{b}}$ and the second one has the order $x^{-1/2}\alpha(x)$ with $\lim\limits_{x\to 0}\alpha(x) = 0$. Since $y(x) \in \mathcal{L}^2$, $c = 0$. The same reasoning shows that $c_1 = 0$. Thus,

$$y(x) = \frac{1}{b}(\sin|x|)^{a/b} \cdot (\cot|x|/2)^{1/b} \int_0^x u(t)(\sin|t|)^{-\frac{a}{b}} \cdot (\sin t)^{-1} \cdot (\tan|t|/2)^{1/b} dt. \tag{1.13}$$

In particular, for $u(x) \equiv 1$ we have

$$y_0(x) \colon = \frac{1}{b}(\sin|x|)^{a/b} \cdot (\cot|x|/2)^{1/b} \int_0^x (\sin|t|)^{-\frac{a}{b}} \cdot (\sin t)^{-1} \cdot (\tan|t|/2)^{1/b} dt.$$

Some elementary estimations show that there are finite limits $\lim\limits_{x\to 0} y_0(x)$, $\lim\limits_{x\to -\pi} y_0(x)$ and $\lim\limits_{x\to\pi} y_0(x)$ with $\lim\limits_{x\to-\pi} y_0(x) = \lim\limits_{x\to\pi} y_0(x)$. Let us show these relations. First, for $t > 0$ we define $w(t) \colon = \big(\frac{\sin t}{t}\big)^{-\frac{a}{b}-1} \cdot \big(\frac{\tan(t/2)}{t}\big)^{1/b}$. Then $\lim\limits_{t\to+0} w(t) = (1/2)^{1/b}$. Moreover, for $x > 0$

$$y_0(x) \colon = \frac{1}{b}(\sin x)^{a/b} \cdot (\cot x/2)^{1/b} \int_0^x w(t) t^{-\frac{a}{b}-1+1/b} dt,$$

so

$$y_0(x) = \frac{1}{b}(\sin x)^{a/b} \cdot (\cot x/2)^{1/b} \frac{b}{1-a} w(\xi_x)(x)^{-\frac{a}{b}+1/b},$$

where $\xi_x \in (0, x)$. The latter yields $y(0) \colon = \lim\limits_{x\to+0} y(x) = \frac{1}{1-a}$. Second, for $t < \pi$ we define $w_+(t) \colon = \big(\frac{\sin t}{\pi - t}\big)^{-\frac{a}{b}-1} \cdot \big(\frac{(\pi-t)}{2} \cdot \tan(t/2)\big)^{1/b}$. Then $\lim\limits_{t\to+\pi-0} w_+(t) = 1$ and for $z(x) \colon = (1+a) \cdot y_0(x) \cdot \big(\sin(x/2)\big)^{\frac{2}{b}}$ we have

$$z(x) = \frac{1+a}{b}(\sin x)^{\frac{a+1}{b}} \cdot \int_0^x w_+(t)(\pi - t)^{-\frac{a}{b}-1-1/b} dt.$$

Let us fix $\epsilon > 0$ Then there is $\delta > 0$ such that $1 - \epsilon < w_+(x) < 1 + \epsilon$ for every $x \in (\pi - \delta, \pi)$. Next, for the same x

$$z(x) = \frac{1+a}{b}(\sin x)^{\frac{a+1}{b}} \cdot \Big(\int_0^{\pi-\delta} w_+(t)(\pi - t)^{-\frac{a}{b}-1-1/b} dt +$$

$$\int_{\pi-\delta}^x w_+(t)(\pi - t)^{-\frac{a}{b}-1-1/b} dt \Big).$$

Since

$$\int_{\pi-\delta}^x w_+(t)(\pi - t)^{-\frac{a}{b}-1-1/b} dt \big) = \frac{b}{1+a} w(\nu_{x,\delta}) \big((\pi - x)^{-\frac{1+a}{b}} - \delta^{-\frac{1+a}{b}}\big)$$

with $\nu_{x,\delta} \in (\pi - \delta, \pi)$,

$$\frac{b}{1+a}(1-\epsilon)\big((\pi - x)^{-\frac{1+a}{b}} - \delta^{-\frac{1+a}{b}}\big) \le \int_{\pi-\delta}^x w_+(t)(\pi - t)^{-\frac{a}{b}-1-1/b} dt \big) \le$$

$$\frac{b}{1+a}(1+\epsilon)\big((\pi - x)^{-\frac{1+a}{b}} - \delta^{-\frac{1+a}{b}}\big).$$

Moreover, for fixed δ

$$\lim_{t\to+\pi-0} \frac{1+a}{b}(\sin x)^{\frac{a+1}{b}} \cdot \int_0^{\pi-\delta} w_+(t)(\pi - t)^{-\frac{a}{b}-1-1/b} dt = 0$$

and

$$\lim_{t\to+\pi-0}(\sin x)^{\frac{a+1}{b}}\cdot w(\nu_{x,\delta})\delta^{-\frac{1+a}{b}}=0,$$

so the equality $\lim\limits_{x\to\pi} y_0(x)=\frac{1}{1+a}$ is practically evident. Note also that $y_0(x)$ is even.

Thus, the function $y_0(x)$ is continuous on $[-\pi,\pi]$ and satisfies the periodic conditions.

Now let $u(x) = c+\int_0^x\phi(t)dt$, where $c = const$ and $\phi(x)\in\mathcal{L}^2$. Then for $x>0$: $|\int_0^x\phi(t)dt|\le x^{1/2}\big(\int_0^x|\phi(t)|^2\big)^{1/2}$. The latter estimation and (1.13) yield $\lim\limits_{x\to 0} y(x)=c\cdot y_0(0)$. The same function can be re-written as following $u(x)=c_+ +\int_\pi^x\phi(t)dt$ or $u(x)=c_- +\int_{-\pi}^x\phi(t)dt$. Then the estimations $|\int_\pi^x\phi(t)dt|\le(\pi-x)^{1/2}\big(-\int_\pi^x|\phi(t)|^2\big)^{1/2}$, $x\in(o,\pi)$ and $|\int_{-\pi}^x\phi(t)dt|\le(\pi+x)^{1/2}\big(\int_{-\pi}^x|\phi(t)|^2\big)^{1/2}$, $x\in(-\pi,o)$ together with Representation (1.13) yield $\lim\limits_{x\to\pi} y(x)=c_+\cdot y_0(\pi)$ and $\lim\limits_{x\to-\pi} y(x)=c_-\cdot y_0(-\pi)$. Thus, $y(x)$ satisfies the periodic condition if and only $u(x)$ satisfies. The latter yields the equality

$$L = SM\,. \tag{1.14}$$

Moreover, we have shown that for every $u(x)=c+\int_0^x\phi(t)dt$ with $\phi(x)\in\mathcal{L}^2$ there is absolutely continuous on $[-\pi,\pi]$ function $y(x)$, such that $(My)(x)=u(x)$, so $\mathfrak{R}(M)$ is dense in $\mathcal{L}^2$ or, equivalently, $\mathrm{Ker}(M^*)=\{0\}$.

Note, that M is boundedly invertible. Indeed, $M=D+iC$, where $C\colon\ \mathcal{L}^2\mapsto\mathcal{L}^2$,

$$(Cy)(x)\colon=i\Big\{\frac{b}{2}\cos x\,y(x)-b\,(\sin x\cdot y(x))'\Big\},\quad \mathfrak{D}(C)=\mathfrak{D}(M)$$

and $D\colon\ \mathcal{L}^2\mapsto\mathcal{L}^2$,

$$(Dy)(x)\colon=\big(1-(a+\frac{b}{2})\,\cos x\big)y(x).$$

Since D is bounded, $\mathfrak{D}(C)=\{y(x)\mid y(x),(Cy)(x)\in\mathcal{L}^2\}$ and a similar reasoning shows that $\mathfrak{D}(C)=\mathfrak{D}(A)$. Thus, if $y(x),z(x)\in\mathfrak{D}(C)$, then for $y(x)$ and $z(x)$ Conditions (1.7) and (1.10) hold true, so

$$\lim_{\epsilon\searrow 0}\sin(\pi-\epsilon)y(\pi-\epsilon)\overline{z(\pi-\epsilon)}=\lim_{\epsilon\searrow 0}\sin(-\pi+\epsilon)y(-\pi+\epsilon)\overline{z(-\pi+\epsilon)}=0.$$

Since C^* is defined by the same differential expression as C, the above equalities show that $\mathfrak{D}(C^*)=\mathfrak{D}(A)$. Thus, C is self-adjoint. Moreover, D is positive and boundedly invertible, so the problem of invertibility of M is equivalent to the problem of regularity of non-real numbers for a self-adjoint operator (for a more detail reasoning see, for instance, [7]).

Now let us prove that L is closed. The operator S restricted to the subspace $\mathcal{L}_1\subset\mathcal{L}^2$ of functions orthogonal to constants has a bounded inverse. Let us find $M^{-1}(\mathcal{L}_1)$. If $(My)(x)\in\mathcal{L}_1$, then $\int_{-\pi}^{\pi}(My)(x)dx=0$, but $\int_{-\pi}^{\pi}(y(x)\sin x)'dx=0$, so $y(x)\in M^{-1}(\mathcal{L}_1)$ if and only if $y(x)\in\mathfrak{D}(M)$ and $\int_{-\pi}^{\pi}\big(1-(a+b)\,\cos x\big)y(x)dx=0$. Let $\mathcal{L}_2\colon=\Big\{\big(1-(a+b)\,\cos x\big)\Big\}^{\perp}$. Since for $y_0(x)$ we have $2\pi=\int_{-\pi}^{\pi}(My_0)(x)dx=\int_{-\pi}^{\pi}y_0(x)\big(1-(a+b)\,\cos x\big)dx$, $y_0(x)\notin\mathcal{L}_2$ and $\mathcal{L}^2=\mathcal{L}_2\dot{+}\{\mu\cdot y_0(x)\}_{\mu\in\mathbb{C}}$. Now let a sequence $\{y_k(x)\}$ be such that $y_k(x)\to y(x)$ and $z_k(x)=(Ly_k)(x)\to z(x)$. Then $(My_k)(x)=v_k(x)+c_k$, where $v_k(x)=((S|_{\mathcal{L}_1})^{-1}z)(x)$, $c_k=const$, $k=1,2,\dots$. Since $(S|_{\mathcal{L}_1})^{-1}$ is bounded, the sequence $\{v_k(x)\}$ has a limite $v(x)$. In turn, in virtue of similar reasons the sequence $w_k(x)=(M^{-1}v_k)(x)$ also has a limite. Simultaneously $y_k(x)=(M^{-1}(v_k+C_k))(x)=w_k(x)+c_ky_0(x)$. Thus, the sequence $\{c_k\}$ has a limite. The rest is straightforward.

□

Corollary 1. *If the parameters a and b satisfy the inequality $2a + b < 2$, then the set $\mathfrak{D}(L)$ is the linear sub-manifold H of the Sobolev space $H^1(-\pi, \pi)$:*

$$\mathfrak{D}(L) = H : f \in H^1(-\pi, \pi), \quad f(\pi) = f(-\pi,) \quad \sin(x)f' \in H^1(-\pi, \pi)$$

and is a Hilbert space with the norm defined as:

$$||f||^2 = ||f||^2_{H^1} + || \sin(x)f'(x)||^2_{H^1}.$$

The reasoning of this corollary is the same as the reasoning of the corresponding proposition in [7].

For the next step we need the following simple remark.

Lemma 1. *Let $\mathcal{H}$ be a Hilbert space, $x_1, x_2 \in \mathcal{H}$, $(x_1, x_2) \neq 0$. Let $\mathcal{H}_1 :\, = \{x_1\}^\perp$, $\mathcal{H}_2 :\, = \{x_2\}^\perp$. Let P_1 and P_2 be ortho-projections onto the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $P_2|_{\mathcal{H}_1}$ is one-to-one mapping onto $\mathcal{H}_2$.*

Proof. Let $\mathcal{H}_3 :\, = \mathcal{H}_1 \cap \mathcal{H}_2$. Without loss of generality we can assume that $\|x_1\| = \|x_2\| = 1$, $(x_1, x_2) = \alpha > 0$. Then $\mathcal{H}_1 = \{\mu \cdot (x_2 - \alpha \cdot x_1)\}_{\mu \in \mathbb{C}} \oplus \mathcal{H}_3$, $\mathcal{H}_2 = \{\mu \cdot (x_1 - \alpha \cdot x_2)\}_{\mu \in \mathbb{C}} \oplus \mathcal{H}_3$. Since $P_2|_{\mathcal{H}_3} = I_{\mathcal{H}_3}$ and $P_2(x_2 - \alpha \cdot x_1) = \alpha \cdot (x_1 - \alpha \cdot x_2)$, the rest is evident.

□

Theorem 2. *If the parameters a and b satisfy the inequality $2a + b < 2$, then the resolvent of L is compact of the Hilbert-Schmidt type.*

Proof. Let $\mathcal{L}_0 \subset \mathcal{L}^2$ be the subspace of constants, $\mathcal{L}_1 :\, = \mathcal{L}_0^\perp$. Since $\mathcal{L}_0 \subset \mathfrak{D}(L)$ and $\mathfrak{R}(L) = \mathcal{L}_1$, the operator L has the following matrix representation

$$L = \begin{bmatrix} 0 & 0 \\ L_{10} & L_{11} \end{bmatrix}$$

with respect to the decomposition $\mathcal{L}^2 = \mathcal{L}_0 \oplus \mathcal{L}_1$, where the operator L_{10}: $1 \to a \cdot \sin x$ is bounded and $\mathfrak{D}(L_{11}) = \{f(x)\,|f \in H^1(-\pi, \pi),\, f(\pi) = f(-\pi),\, \sin(x)f'(x) \in H^1(-\pi, \pi),\, \int_{-\pi}^{\pi} f(x)dx = 0\}$. Let us analyze the properties of L_{11}. From Theorem 1 we have $L_{11} = SM|_{\mathcal{L}_1}$. Let $\mathcal{L}_2 = M(\mathfrak{D}(M) \cap \mathcal{L}_1)$, $y_0(x) = (M^{-1}1)(x)$, $z_0(x) = ((M^*)^{-1}1)(x)$. Then for $y(x) \in \mathfrak{D}(M) \cap \mathcal{L}_1$ we have $0 = (y, 1) = (y, M^* z_0) = (My, z_0)$, so $\mathcal{L}_2 = \{z_0\}^\perp$. From the other hand, $(1, z_0) = (My_0, z_0) = (y_0, M^* z_0) = (y_0, 1)$. Since (see the proof of Theorem 1) $(y_0, 1) \neq 0$, the pair $\{1, z_0\}$ is under the conditions of Lemma 1. Let P_1 be the ortho-projection onto $\mathcal{L}_1$. Then $L_{11} = S \cdot (P_1|_{\mathcal{L}_2}) \cdot (M|_{\mathcal{L}_1})$, so $L_{11}^{-1} = (M^{-1}|_{\mathcal{L}_2}) \cdot (P_1|_{\mathcal{L}_1})^{-1} \cdot (S|_{\mathcal{L}_1})^{-1}$. Thus, L_{11}^{-1} is an operator of the Hilbert-Schmidt type. Since

$$R_\lambda(L) = \begin{bmatrix} R_\lambda(0) & 0 \\ \frac{1}{\lambda} R_\lambda(L_{11}) L_{10} & R_\lambda(L_{11}) \end{bmatrix},$$

the rest is straightforward.

□

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О ФАКТОРИЗАЦИИ ОДНОГО ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА, ВОЗНИКАЮЩЕГО В ГИДРОДИНАМИКЕ

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Спектральные свойства линейных операторов играют важную роль в анализе устойчивости динамических систем. В заметке исследуются свойства несамосопряженного дифференциального оператора второго порядка, связанного с исследованием проблемы устойчивости стационарного динамического состояния тонкой пленки, образованной вязкой ньютоновской жидкостью и расположенной на внутренней поверхности вращающегося цилиндра, при наличии гравитационного поля. Линеаризация по малому параметру (отношению толщины потока к размеру цилиндра) в этом случае порождает дифференциальный оператор с областью определения, вложенной в прямую сумму двух подпространств, натянутых, соответственно, на базисы $\{e^{inx}\}$ и $\{e^{-inx}\}$ $(n > 0)$, причем указанные подпространства не являются инвариантными по отношению к оператору, и одномерного подпространства констант. Доказывается, что этот оператор допускает представление в виде произведения двух дифференциальных операторов первого порядка. Полученное представление используется для доказательства компактности резольвенты исследуемого оператора и непосредственного описания его области определения.

Ключевые слова: спектральный анализ дифференциального оператора, факторизация, гидродинамика, прямое/обратное уравнение теплопроводности.

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