КРАТКИЕ СООБЩЕНИЯ

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COMPUTATIONAL EXPERIMENT FOR ONE MATHEMATICAL MODEL OF ION-ACOUSTIC WAVES

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In the article the mathematical model of ion-acoustic waves in a plasma in an external magnetic field is studied. This model can be reduced to a Cauchy problem for a Sobolev type equation of the fourth order with polynomially (A,p)-bounded operator pencil. Therefore abstract results on solvability of the Cauchy problem for such equation can be used. In the article a theorem on the unique solvability of the Cauchy – Dirichlet problem is mentioned. Based on the theoretical results there was developed an algorithm for the numerical solution of the problem, using a modified Galerkin method. The algorithm is implemented in Maple. The article includes description of this algorithm. It is illustrated by model examples showing the work of the developed program.

Keywords: mathematical model; ion-acoustic waves; Galerkin method.

Introduction. Consider equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2}+\omega^2_{B_i}\right)(\Delta_3\Phi-\frac{1}{r_D^2}\Phi)+\omega^2_{p_i}\frac{\partial^2}{\partial t^2}\Delta_3\Phi+\omega^2_{B_i}\omega^2_{p_i}\frac{\partial^2\Phi}{\partial x_3^2}=0, \tag{1}$$

first obtained by Y.D. Pletner [2], which describes the ion-acoustic waves in a plasma in an external magnetic field. Here Δ_3 is a Laplace operator in $\mathbb{R}^3$, the function Φ is a generalized potential of the electric field, the constants $\omega^2_{B_i}$, $\omega^2_{p_i}$ and r_D^2 characterize ion gyrofrequency, Langmuir frequency and the Debye radius, respectively. Transform equation (1) and consider a more general problem.

Let $\Omega=(0,a)\times(0,b)\times(0,c)\subset\mathbb{R}^3$. In the cylinder $\Omega\times\mathbb{R}$ consider the Cauchy – Dirichlet problem

$$\begin{gathered} v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x),\\ \frac{\partial^2 v}{\partial t^2}(x,0)=v_2(x),\quad \frac{\partial^3 v}{\partial t^3}(x,0)=v_3(x),\quad x\in\Omega \end{gathered} \tag{2}$$

$$v(x,t)=0,\quad (x,t)\in\partial\Omega\times\mathbb{R} \tag{3}$$

for equation

$$(\Delta-\lambda)\frac{\partial^4 v}{\partial t^4}+(\Delta-\lambda')\frac{\partial^2 v}{\partial t^2}+\alpha\frac{\partial^2 v}{\partial x_3^2}=0, \tag{4}$$

describing the ion-acoustic waves in a plasma in a magnetic field, and the negative values of the parameter λ do not contradict the physical meaning of this problem. Stochastic mathematical model of ion-acoustic waves in a plasma was considered in [3].

1. Analytical Study of the Mathematical Model of Ion-Acoustic Waves in a Plasma in a Magnetic Field. Introduce the eigenfunctions of the Laplace operator Δ in the domain Ω satisfying conditions (3): $\varphi_{kmn} = \left\{\sin\frac{\pi k x_1}{a}\sin\frac{\pi m x_2}{b}\sin\frac{\pi n x_3}{c}\right\}$, where k, m, $n \in \mathbb{N}$, and the eigenvalues $\lambda_{kmn} = -(k^2+m^2+n^2)$. Obviously, the spectrum $\sigma(\Delta)$ is negative, discrete with finite multipicities and thickens only to $-\infty$. Since $\{\varphi_k\} \subset C^\infty(\Omega)$, then

$$\mu^4 A - \mu^3 B_3 - \mu^2 B_2 - \mu B_1 - B_0 =$$

$$\sum_{k,m,n=1}^{\infty} [(\lambda_{kmn}-\lambda)\mu^4 + (\lambda_{kmn}-\lambda')\mu^2 - \alpha(\frac{\pi n}{c})^2] < \varphi_{kmn}, \cdot > \varphi_{kmn},$$

where $< \cdot, \cdot >$ is a scalar product in $L^2(\Omega)$.

Lemma 1. [4] *(i) Let $\lambda \notin \sigma(\Delta)$. Then the pencil $\vec{B}$ is polynomially $(A,0)$-bounded.*

(ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$. Then the pencil $\vec{B}$ is polynomially $(A,1)$-bounded.

(iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda')$. Then the pencil $\vec{B}$ is polynomially $(A,3)$-bounded.

Theorem 1. [4] *(i) Let $\lambda \notin \sigma(\Delta)$. Then, for arbitrary $v_0, v_1, v_2, v_3 \in \mathfrak{U}$ there exists a unique solution of problem (2) – (4).*

(ii) Let $\lambda \in \sigma(\Delta)$ и $\lambda = \lambda'$. Then for arbitrary $v_0, v_1, v_2, v_3 \in \mathfrak{U}^1$, i.e., such that

$$\sum_{\lambda_{kmn}=\lambda} < \varphi_{kmn}, v_j >= 0, \; j = 0, ..., 3,$$

there exists a unique solution of problem (2) – (4).

2. Numerical Solution Algorithm. Based on the theoretical results there was developed an algorithm for numerical solution of problem (2) – (4) modelling ion-acoustic waves in a plasma in an external magnetic field, implemented in a software environment Maple 15.0. The program uses a phase space method and a modified Galerkin method.

A numerical solution algorithm is shown in a block diagram in picture 1. The developed program allows you to:

1. Specify the sizes of the domain Ω for the mathematical model of ion-acoustic waves in a plasma in an external magnetic field.

2. Enter the parameters of the equation: $\lambda, \lambda', \alpha$; initial data: $v_0(x,y,z), v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)$, and the order of Galerkin approximations N.

3. Print the numerical solution of the problem.

4. Get a graphical image of the received waves with animated distribution over time.

A detailed description of the algorithm (each block of the algorithm corresponds to one step):

Step 1. After the start of the program the number of terms in a Galerkin sum N, parameters λ, λ_1, α, initial data v_0, v_1, v_2, v_3, the positive numbers a, b, c and period $\tau : t \in [0, \tau]$ are entered.

Step 2. In a cycle approximate solution V is represented as the Galerkin sum $\sum\limits_{i,j,k=1}^{N} A_{i,j,k}(t)\sin\frac{\pi ix}{a}\sin\frac{\pi jy}{b}\sin\frac{\pi kz}{c}$.

Step 3. Expression for V is substituted into equation.

Step 4. Start the cycle by i,j,k from 1 to N.

Step 5. Taking the inner product of equation by the corresponding eigenfunctions $\varphi_i(x), \psi_j(y), \chi_k(z)$.

Step 6. Checking if λ belongs to the spectrum of the Laplace operator.

If sixth step is true:

Step 7. Verification of condition $\lambda = \lambda_1$.

If seven step is true:

Step 8. Solving of an algebraic equation with respect to $A_{i,j,k}(t)$.

If seven step is false:

Step 9. Initial data v_0, v_1 are multiplied by the eigenfunctions $\varphi_i(x), \psi_j(y), \chi_k(z)$.

Step 10. Solving of the ordinary differential equation of the second order, corresponding to the current number i,j,k in the cycle.

If the sixth step false:

Step 11. Initial data v_0, v_1, v_2, v_3 are scalar multiplied by the eigenfunctions $\varphi_i(x), \psi_j(y), \chi_k(z)$.

Step 12. Solving of the ordinary differential equation of the fourth order corresponding to the current number i,j,k.

Step 13. End of cycle by i,j,k.

Step 14. Founded Galerkin coefficients $A_{i,j,k}(t)$ are substituted into the approximate solution obtained in step 3.

Step 15. The resulting approximate solution is displayed as a graph of the solution with the animation over time from 0 to τ, with chosen fixed variable (for example z).

3. Numerical Experiment. Illustrate the described algorithm by several computational examples.

Example 1. Consider the problem

$$v(x,y,z,t) = 0, \quad (x,y,z,t) \in \partial\Omega \times \mathbb{R}, \tag{5}$$

$$\begin{gathered} v(x,y,z,0) = \sin x \sin y \sin z, \quad v_t(x,y,z,0) = 10 \sin x \sin y \sin z, \\ \frac{\partial^2 v}{\partial t^2}(x,y,z,0) = 3 \sin x \sin y \sin z, \quad \frac{\partial^3 v}{\partial t^3}(x,y,z,0) = \sin x \sin y \sin z, \end{gathered} \tag{6}$$

$$(\Delta - 2)\frac{\partial^4 v}{\partial t^4} + (\Delta - 1)\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial x_3^2} = 0. \tag{7}$$

It is required to find the numerical solution of problem (5) – (7) when $\lambda = 2, \lambda' = 1, \alpha = 1$, in a domain $[0,\pi] \times [0,\pi] \times [0,\pi]$, $t \in [0,2]$.

Eigenfunctions $\varphi_i(x), \psi_j(y), \chi_k(z)$ of the homogeneous Dirichlet problem for the Laplace operator in the domain $[0,\pi] \times [0,\pi] \times [0,\pi]$ have the form $\{\sin ix, \sin jy, \sin kz\}$. Obviously, in this case, equation (7) is not degenerate, therefore, the algorithm will take place in accordance with steps 11, 12 described in Section 2 of this article.

Graph of the solution is presented in picture 2 a.

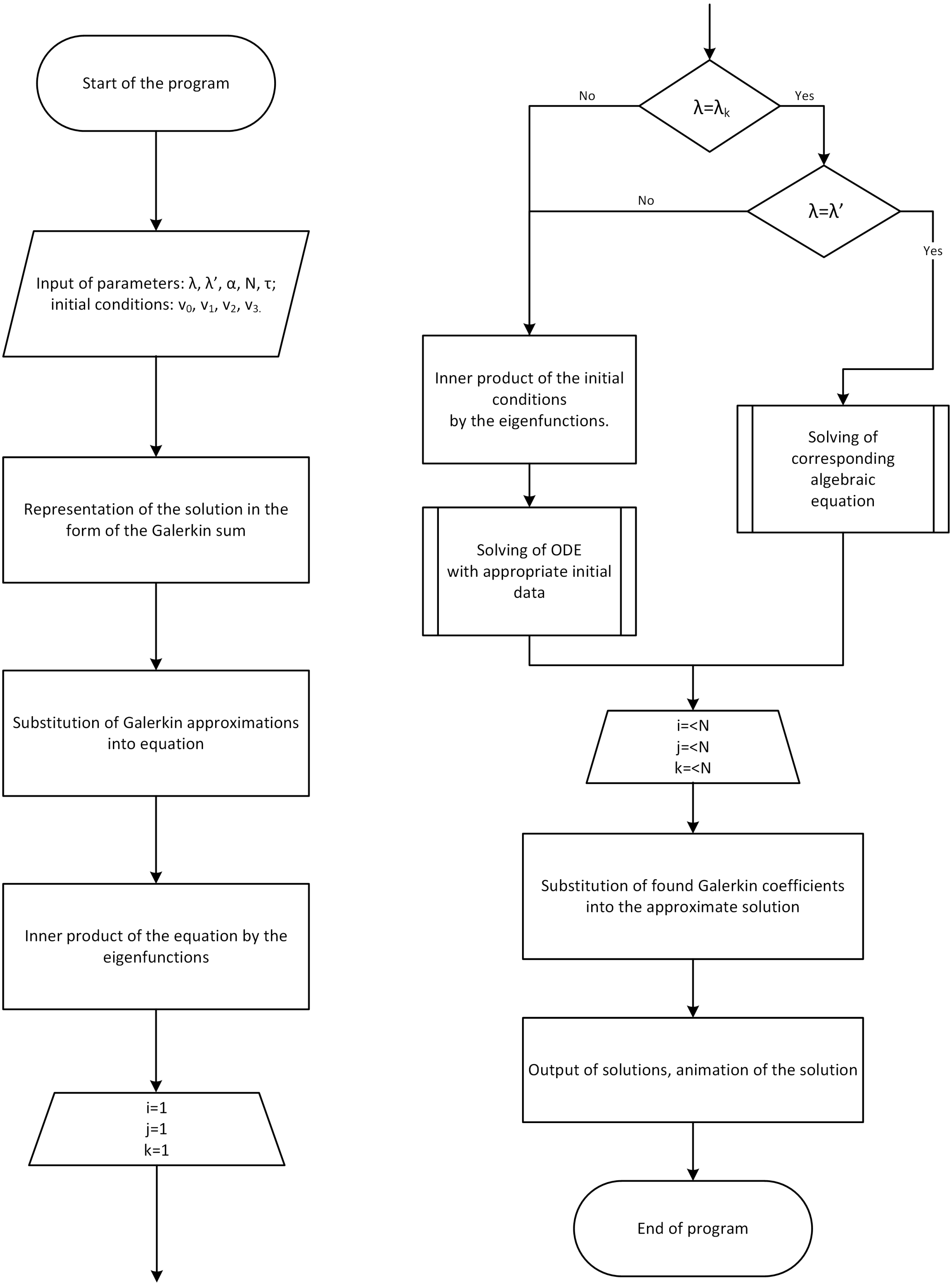


Fig. 1. A block diagram of algorithm

Example 2. Consider the problem

$$v(x,y,z,t) = 0, \quad (x,y,z,t) \in \partial\Omega \times \mathbb{R}, \tag{8}$$

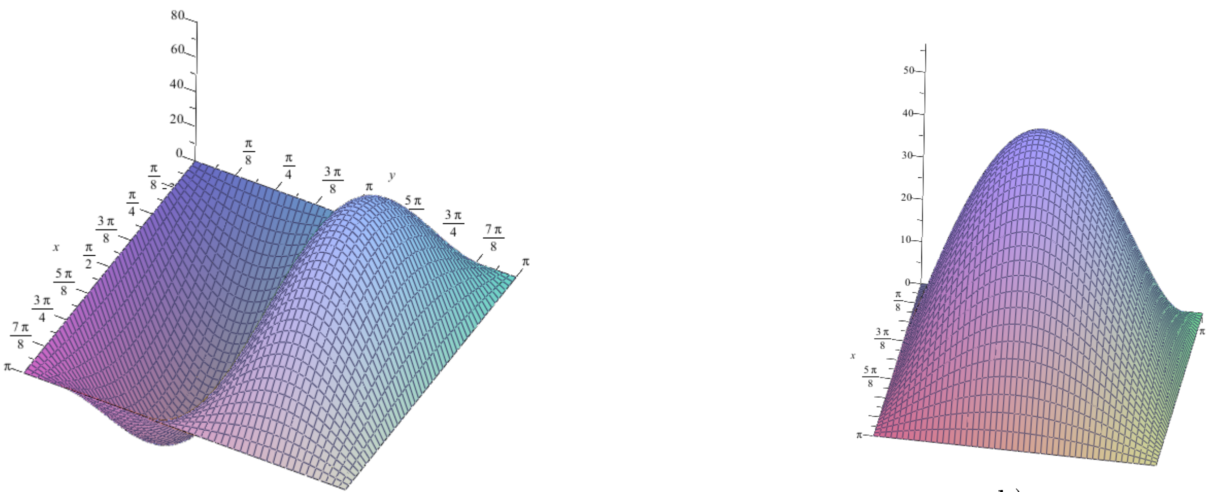
$$\begin{gathered} v(x,y,z,0) = 8\sin x \sin y \sin z, \quad v_t(x,y,z,0) = 0{,}1 \sin x \sin y \sin z, \\ \frac{\partial^2 v}{\partial t^2}(x,y,z,0) = 5\sin x \sin y \sin z, \quad \frac{\partial^3 v}{\partial t^3}(x,y,z,0) = \sin x \sin y \sin z, \end{gathered} \tag{9}$$

$$(\Delta + 4)\frac{\partial^4 v}{\partial t^4} + (\Delta + 4)\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x_3^2} = 0. \tag{10}$$

It is required to find the numerical solution of problem (7) – (10) when $\lambda = 4, \lambda' = 4, \alpha = 1$, in a domain $[0,\pi] \times [0,\pi] \times [0,\pi]$, $t \in [0,3]$.

Eigenfunctions $\varphi_i(x), \psi_j(y), \chi_k(z)$ of the homogeneous Dirichlet problem for the Laplace operator in the domain $[0,\pi] \times [0,\pi] \times [0,\pi]$ have the form $\{\sin ix, \sin jy, \sin kz\}$. Obviously, in this case, equation (10) is degenerate, therefore, the algorithm will take place in accordance with step 8 described in Section 2 of this article.

Graph of the solution is presented in picture 2 b.



a) b)

Fig. 2. a) Solution from example 1; b) solution from example 2

References

1. Zamyshlyaeva A.A. *Lineynye uravneniya sobolevskogo tipa vysokogo poryadka* [Linear Sobolev Type Equations of High Order]. Chelyabinsk, Publ. Center of the South Ural State University, 2012.
2. Sveshnikov A.G., Al'shin A.B., Korpusov M.O., Pletner Yu.D. *Lineynye i nelineynye uravneniya sobolevskogo tipa* [Linear and Non-Linear Equations of Sobolev Type]. Moscow, FIZMATLIT, 2007.
3. Zamyshlyaeva A.A. [Stochastic Mathematical Model of Ion-Acoustic Waves in a Plasma]. *Estestvennye i tekhnicheskie nauki* [Natural and Technical Sciences], 2013, no. 4, pp. 284–292. (in Russian)

4. Zamyshlyaeva A.A. [A Mathematical Model of Ion-Acoustic Waves in a Plasma in a Magnetic Field]. *Materialy mezhdunarodnoy konferentsii "Voronezhskaya zimnyaya matematicheskaya shkola S.G. Kreyna – 2014"*, 2014, pp. 142–144. (in Russian)

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ВЫЧИСЛИТЕЛЬНЫЙ ЭКСПЕРИМЕНТ ДЛЯ ОДНОЙ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ИОННО-ЗВУКОВЫХ ВОЛН

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В статье рассмотрена математическая модель ионно-звуковых волн в плазме во внешнем магнитном поле. Данная математическая модель может быть редуцирована к задаче Коши для уравнения соболевского типа четвертого порядка с полиномиально (A,p)-ограниченным пучком операторов. Следовательно применимы абстрактные результаты по разрешимости задачи Коши для такого уравнения. В статье сформулирована теорема об однозначной разрешимости задачи Коши – Дирихле. На основе теоретических результатов был разработан алгоритм для численного решения задачи, основанный на модифицированном методе Галеркина. Алгоритм реализован в среде Maple. В конце приведены примеры, в которых решение получено при помощи разработанной программы.

Ключевые слова: математическая модель; ионно-звуковые волны; метод Галеркина.

References

1. Замышляева, А.А. Линейные уравнения соболевского типа высокого порядка / А.А. Замышляева. – Челябинск: Издательский центр ЮУрГУ, 2012.
2. Свешников, А.Г. Линейные и нелинейные уравнения соболевского типа / А.Г. Свешников, А.Б. Альшин, М.О. Корпусов, Ю.Д. Плетнер. – М.:ФИЗМАТЛИТ, 2007.
3. Замышляева, А.А. Стохастическая математическая модель ионно-звуковых волн в плазме / А.А. Замышляева // Естественные и технические науки. – 2013. – № 4. – С. 284–292.
4. Замышляева, А.А. Математическая модель ионно-звуковых волн в плазме во внешнем магнитном поле / А.А. Замышляева, А.С. Муравьев // Материалы международной конференции «Воронежская зимняя математическая школа С.Г. Крейна – 2014». – Воронеж, 2014. – С. 142–144.

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